1.1 Using the definition of acceleration:
\[
\alpha = \frac{\Delta v}{\Delta t} = \frac{60 \text{ mph} - 0}{9.2 - 0} = \frac{88 \text{ ft/sec} - 0}{9.2 \text{ sec}} = 9.57 \text{ ft/sec}^2
\]

1.2 Differentiate \( x(t) \) to obtain the velocity:
\[
v(t) = \dot{x}(t) = -10t + 88 \text{ ft/sec}.
\]
Differentiating again yields the acceleration:
\[
a(t) = \ddot{x}(t) = -10 \text{ ft/sec}^2.
\]
So \( v(t) = 0 = -10t + 88 \).
Solving this for \( t \) yields that:
\[
v(t) = 0 \text{ at } t = 8.8 \text{ sec}.
\]

1.3 Evaluating \( x \) at zero yields \( x(0) = 5 \text{ m} \).
Differentiating yields \( \dot{x}(t) = v(t) = 3t^2 - 2 \)
so that \( v(0) = -2 \text{ m/s} \).
Likewise
\[
a(t) = \ddot{x}(t) = 6t
\]
so that \( a(0) = 0 \).
Now at \( t = 3 \text{ sec} \),
\[
x(3) = 27 - 2(3) + 5 = 26 \text{ m}.
\]
\[
v(3) = 27 - 2 = 25 \text{ m/s}
\]
and \( a(3) = 18 \text{ m/s}^2 \).
Since the velocity changes sign during this interval, the particle has doubled back and to compute the total distance traveled during the interval you must compute how far it travels before it changes direction and then add this to the distance traveled after the particle has changed direction. The particle changes direction when the velocity is zero, or at the value of \( t \) for which
\[
v(t) = 3t^2 - 2 = 0,
\]
or at time
\[
t = 0.8165.
\]
The particle first moves from \( x(0) = 5 \text{ m} \) to \( x(0.8165) = 3.9 \text{ m} \) or a distance of 1.1 m.
It then changes direction and moves from 3.9 m to $x(3) = 26$ m. Thus it travels a total distance of $(26 - 3.9) + 1.1 = 1.1 + 1.1 + 21 = 23.2$ m.

1.4 This again is straightforward differentiation:

$v(t) = \dot{x}(t) = 2t - 2$.

This is zero when $t$ satisfies:

$2t - 2 = 0$ or $t = 1$ sec.

Next:

$a(t) = \ddot{x}(t) = \dot{v}(t) = 2 \text{ ft/sec}^2$ which is constant acceleration.

Alternately, the distance traveled can be computed directly from integrating the absolute value of the velocity:

$d = \int_0^3 |3t^2 - 2| dt = 23.178$ m

1.5 Solution: $v(t) = \dot{x}(t) = 6 \cos 2t$ m/s. $a(t) = \ddot{x}(t) = \dot{v}(t) = -12 \sin(2t)$ m/sec$^2$.

Setting $-12 \sin(2t) = 0$, yields $2t = \pi$, or $t = 0, \pi/2, \pi, 3\pi/2...n\pi/2$, for the times for the acceleration to hit zero.

1.6 From the definition, straightforward differentiation yields: $x_A = (3t^2 + 6t)$ ft

so the $v_A = (6t + 6)$ ft/sec

and $a_A = 6$ ft/sec$^2$. Likewise, $x_B = 3t^3 + 2t$

so that $v_B = (9t^2 + 2)$ ft/sec

and $a_B = 18t$ ft/sec$^2$

a) Thus at $t = 1$ sec:

$x_A(1) = 9$ ft, $v_A(1) = 12$ ft/sec, and $a_A(1) = 6$ ft/sec$^2$

$x_B(1) = 5$ ft, $v_B(1) = 11$ ft/sec, and $a_B(1) = 18$ ft/sec$^2$

So that $A$ is ahead of $B$ and has the largest velocity, but $B$ is accelerating faster than $A$.

b) Now at $t = 2$ sec:

$x_A(2) = 24$ ft, $v_A(2) = 18$ ft/sec, and $a_A(2) = 6$ ft/sec$^2$

$x_B(2) = 28$ ft, $v_B(2) = 38$ ft/sec, and $a_B(2) = 36$ ft/sec$^2$

Now $B$ is ahead of $A$, and has larger velocity and acceleration.
c) To find where $A$ and $B$ have moved the same distance, let $x_A(t) = x_B(t)$ and solve for $t$. This yields

$$3t^2 + 6t = 3t^3 + 2t, \text{ or } t^2 - t - 4/3 = 0 \text{ and } t = 0$$

Solving yields $t = +1.758s$ and $t = -1.758s$.

Here we are interested in the positive value of time so that

$$x_A(1.758) = x_B(1.758) = 3(1.758)^2 + 6(1.758) = 3(1.758)^3 + 2(1.758) = 19.81 \text{ ft}.$$  

1.7 Note that this is an inverse problem. Straightforward differentiation yields:

$$v(t) = 3(-e^{-t} \sin 10t + e^{-t}10 \cos 10t) = 3e^{-t}(10 \cos 10t - \sin 10t)$$

$$a(t) = 3e^{-t}(-10^2 \sin 10t - 10 \cos 10t) - 3e^{-t}(10 \cos 10t - \sin 10t)$$

$$= -3e^{-t}(99 \sin 10t - 20 \cos 10t)$$  

1.8 From the definition

$$x(t) = t^3 - 6t^2 - 15t + 40 \text{ ft}$$

so that:

$$\dot{x} = v = 3t^2 - 12t - 15 \text{ ft/sec},$$

and $\ddot{x} = a = 6t - 12 \text{ ft/sec}.$

a) $v(t) = 0$ requires $3t^2 - 12t - 15 = 0$

or $t^2 - 4t - 5 = 0$.

Solving for $t$ yields:

$t = -1$ and $5$.

Taking the positive value of $t$, the velocity is zero at $t = 5 \text{ sec}.$

b) At $t = 5 \text{ sec}$,

$$x(t) = x(5) = (5)^3 - 6(5)^2 - (15)(5) + 40 = -60 \text{ ft}.$$  

At rest $t = 0$,

$$x(0) = 40.$$  

Then the particle has moved from 40 to -60 or $40 + 60 = 100 \text{ ft}.$

c) $a(5) = (6)(5) - 12 = 18 \text{ ft/sec}^2.$

1.9 Note: The purpose of this problem is to hit home the idea that the distance traveled and the displacement are different. This problem is easiest to solve using computational software as it involves plots. Students could also use a symbolic processor to compute the derivatives, although it would be a little over
kill and they must be reminded that simple derivations should be something they can do in their head, on tests, while complicated derivatives are more accurately done with software. The plots are also easy to sketch by hand, but if they are inexperienced at plotting using software, it is best to start them off with some simple plots.

a) \(v(t) = \dot{x} = 0.6 \cos 2t\)
\(a(t) = \ddot{x} = -1.2 \sin 2t\).

b) \(3 \sin 2t\) travels a distance of 0.3m in the time 0 to \(\frac{\pi}{4}\) sec and back another 0.3m returning back to the origin from \(\frac{\pi}{4}\) to \(\frac{\pi}{2}\) s. So the total distance traveled is \(0.3 + 0.3 = 0.6\text{m}\) (maybe look at the plot first).

c) The position of the mass at \(t = \frac{\pi}{2}\) s however is \(x\left(\frac{\pi}{2}\right) = 0.3 \sin\left(\frac{2\pi}{2}\right) = 0\).

Note the position at any point is not always the distance traveled, which in b) is shown to be 0.6m.
1.10 Differentiation yields $v(t) = 88.92 \sin 0.26t$ ft, and $a(t) = 23.12 \cos 0.26t$ ft/sec². 
The plot of each follows.

\[ t = 0, 0.1 \ldots 12.2 \]
\[ x(t) = 342 \cdot (1 - \cos (0.26 \cdot t)) \quad v(t) = 88.92 \cdot \sin (0.26 \cdot t) \]
\[ a(t) = 23.12 \cdot \cos (0.26 \cdot t) \]

**FIGURE S1.10**

1.11 The following code in Matlab computes the velocity from the displacement data and plots it:
\[
x = [8 \ 9 \ 11 \ 13 \ 14 \ 15 \ 17 \ 18 \ 22 \ 27 \ 32 \ 37 \ 41 \ 44 \ 46 \ 48 \ 49 \ 49 \ 48 \ 47 \ 46];
\]
\[
t = 0; 0.01:0.02;
\]
\[
n = \text{length} (x);
\]
\[
v = 0* x;
\]
\[
dt = .01;
\]
\[
\text{for } i = 1:n-1
\]
\[
v(i+1) = (x(i+1)-x(i))/dt;
\]
\[
\text{end}
\]
\[
v
\]
\[
\text{plot(t,v),xlabel('t*dt or elapsed time'), title('velocity versus time')}
\]
This produces the following output:

\[ v = \]

Columns 1 through 12
0 100 200 200 100 100 200 100 400 500 500 500
Columns 13 through 21
400 300 200 200 100 0 -100 -100 -100
And the following plot:

![Plot](image)

**FIGURE S1.11a**

The following produces the corresponding acceleration:

```matlab
EDU >> a = 0 * v;
EDU >> for i = 1:n-1
    a(i+1) = (v(i+1) - v(i))/dt;
end
EDU > a
```

a=

Columns 1 through 6
0 10000 10000 0 -10000 0
Columns 7 through 12
10000 -10000 30000 10000 0 0
Columns 13 through 18
-10000 -10000 -10000 0 -10000 -10000
1.12 Consider the following Matlab code which uses the central difference to compute the velocity:

\begin{verbatim}
x=[8 9 11 13 14 15 17 18 22 27 32 37 41 44 46 48 49 49 48 47 46];
t=0:.01:0.2;
n=length(x);
v=0*x;
dt=.01;
for i=1:n-1
  v(i)=(x(i+1)-x(i-1))/(2*dt);
end
v
plot(t,v),xlabel('t*dt or elapsed time'),title('velocity versus time')
\end{verbatim}
This results in the following values:

\[ v = \]

Columns 1 through 12
0 150 200 150 0 100 150 150 250 0 450 500 500 450
Columns 13 through 21
350 250 200 150 50 -50 -100 -100 0

And the following plot:

The following is the Mathcad code for solving this problem:

\[ i := 1 .. 19 \quad \Delta t = 0.01 \]

\[ v_{i+1} := \frac{x_{i+1} - x_{i-1}}{2 \Delta t} \]
1.13 Solution:
\[ y(t) = -4.905t^2 + 20t \text{ (m)} \]
\[ v = \dot{y}(t) = -9.81t + 20 \text{ (m/s)} \]
\[ a = \ddot{v} = -9.81 \text{ (m/s)} \]
The position, velocity and acceleration at \( t = 5 \) are
\[ y(5) = -22.625 \text{ m}, \ v(5) = -29.05 \text{ m/s}, \ a(5) = -9.81 \text{ m/s}^2 \]
Note that the ball returns to its initial state of \( y(5) = 0 \) when \( t \) satisfies
\[ -4.905t^2 + 20t = 0 \text{ or, } t = 20/4905 = 2.07 \text{ sec.} \]
Then to obtain the total distance traveled by the ball, we need to calculate when the ball changes direction, i.e., when \( v(t) = 0 \).
\[ v(t) = 0 = -9.81t + 20 \text{ or } t = 2.0395. \]
From \( t = 0 \) to 2.039 sec. the ball travels a distance of \( y = 20.39 \text{ ft.} \)
It then travels back past zero (the top of the building) another 20.39 ft. to \( y(0) = 0 \).
It then travels on a distance of \( y(5) = -22.625 \), beyond zero.
Thus the total distance traveled by the ball is \( 20.39 + 20.39 + 22.625 = 63.4 \text{ m.} \)

1.14 Solution: \( x, v \) and \( a \) are given respectively by:
\[ x(t) = e^{-ct} \sin \omega t \]
\[ v(t) = -ce^{-ct} \sin \omega t + e^{-ct} \omega \cos \omega t \]
\[ a(t) = c^2 e^{-ct} \sin(\omega t) - 2c \omega e^{-ct} \cos \omega t - \omega^2 e^{-ct} \sin \omega t \]
\[ x(0) = 0, \ v(0) = \omega, \ a(0) = -2c\omega \]

1.15 Solution:
\[ x(t) = 3t^3 - 2t^2 + 5, \ x(0) = 5m \]
\[ v(t) = \dot{x}(t) = 9t^2 - 4t, \ v(0) = 0 \text{ m/s} \]
\[ a(t) = \ddot{v}(t) = 18t - 4, \ a(0) = -4 \text{ m/s}^2 \]

1.16 The velocity and acceleration are respectively:
\[ v(t) = 4 \cdot t \cdot \cos(\pi \cdot t) - 2 \cdot t^2 \cdot \sin(\pi \cdot t) \cdot \pi \]
\[ a(t) = 4 \cdot \cos(\pi \cdot t) - 8 \cdot t \cdot \sin(\pi \cdot t) \cdot \pi - 2 \cdot t^2 \cdot \cos(\pi \cdot t) \cdot \pi^2 \]
so that \( x(0) = v(0) = 0 \) and \( a(0) = 4 \)
1.17 Solution:
\[ y(t) = 3t^2 - 20, \text{ so } y(0) = -20 \text{ m} \]
\[ v(t) = \dot{y}(t) = 6t, \text{ so } \dot{v}(0) = 0 \text{ m/s} \]
\[ a(t) = \ddot{y}(t) = 6, \text{ so } a(0) = 6 \text{ m/s}^2 \]

1.18 Solution:
\[ x(t) = \exp(-0.1 \cdot t) \cdot (3 \cos t \cos(2 \cdot t) + \sin(2 \cdot t)), \ x(0) = 3 \]
\[ v(t) = 1.7 \cdot \exp(-1 \cdot t) \cdot \cos(2 \cdot t) - 6.1 \cdot \exp(-1 \cdot t) \cdot \sin(2 \cdot t), \ v(0) = 1.7 \]
\[ a(t) = -12.36 \exp(-0.1t) \cdot \cos(2t) - 2.79 \cdot \exp(-0.1t) \cdot \sin(2t), \ a(0) = -12.37 \]

1.19 Solution:
\[ x(t) = 5 \cdot t - \exp(-t) \cdot 3 \cdot t, \ x(0) = 0; \]
\[ v(t) = 5 + 3 \cdot \exp(-t) \cdot t - 3 \cdot \exp(-t), \ v(0) = 2; \]
\[ a(t) = -3 \cdot \exp(-t) \cdot t + 6 \cdot \exp(-t), \ a(0) = 6. \]

1.20 Solution: There is no derivative at \( t = 5 \), however the problem may be split using heaviside functions and differentiated over the two intervals.

\[ t = 0, .01 .. 10 \]
\[ y(t) = 5 \cdot \Phi(5 - t) - (25 - 20 \cdot \sin(\pi \cdot t - \pi)) \cdot \Phi(t - 5) \]
\[ y1(t) = 5 \cdot \Phi(5 - t) - 20 \cdot \cos(\pi \cdot t - \pi) \cdot \Phi(t - 5) \] is the velocity
\[ y11(t) = (20 \cdot \pi^2 \cdot \sin(\pi \cdot t - \pi)) \cdot \Phi(t - 5) \] is the acceleration

**FIGURE S1.20**
During the 1st interval the slope \(dy/dt\) is constant. From the plot \(y(t) = \frac{20-0}{4-0} t - 5t\) for \(t = 0\) to 5 sec. In the interval \(t > 5\), \(y(t) = 25 + A \sin(\omega t + \phi)\) where \(\omega\) is the frequency and \(\phi\) is the phase and \(A\) is the amplitude of the sine wave.

From the plot \(A = 20\), the period \(T = 2\) sec, so that \(\omega = \frac{2\pi}{T} = \pi\). To find the phase evaluate \(y(5) = 25 = 25 + 20(\sin(\pi t + \phi))\) so that \(\pi t + \phi = n\pi\), where \(n\) is any integer or \(\phi = (n - 5)\pi\), so \(\phi = \pi\) will work. Thus:

\[
\begin{align*}
y(t) &= \begin{cases} 5t & 0 < t < 5 \\ 20 \sin(\pi t - \pi) & t > 5 \end{cases} \\
y'(t) &= \begin{cases} 5 & 0 < t < 5 \\ 20\pi \cos(\pi t - \pi) & t > 5 \end{cases} \\
y''(t) &= \begin{cases} 0 & 0 < t < 5 \\ -20\pi^2 \sin(\pi t - \pi) & t > 5 \end{cases}
\end{align*}
\]

1.21 Solution: \(a_{ave} = \frac{60 - 0}{5} \text{ mph} = \frac{60}{5} \text{ mph} = \frac{12 \times 5280 \text{ ft}}{\text{hour} \times 3600 \text{ sec}} = \frac{(12)(5280)}{300} \text{ ft/sec}^2 = 17.6 \text{ ft/sec}^2\)

\[
17.6 \text{ ft/sec}^2 = 17.6 \frac{\text{m}}{\text{sec}^2} = 5.4 \text{ m/sec}^2
\]

\[
\frac{5.4 \text{ m/sec}}{9.81 \text{ m/sec}^2} = 0.55 \text{ g’s or about 55% of a g.}
\]

1.22 One answer: a 4 min mile is near a record speed for trained runners so:

\[
v = \frac{1 \text{ mile}}{4 \text{ min}} = \frac{1 \text{ mile}}{4 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \cdot \frac{5280 \text{ ft}}{\text{mile}} = 22 \text{ ft/sec}
\]

or about 6.71 meter/s, or about 15 mph. On the other hand a sprinter can cover 100 m dash in 10 seconds, or 10 m/s.

1.23 The Matlab solution follows (see problem 1.11 and 1.12 also):

\[
\begin{align*}
\text{v} &= [0.2 0.27 0.3 0.3 0.3 0.3 0.3 0.3 0.35 0.4 0.5]; n=\text{length(v)}; \\
\text{t} &= 0:0.1:11; \\
\text{dt} &= 0.1; \text{a} = 0.*\text{v}; \text{x} = 0.*\text{v}; \% \text{zeros in a and x} \\
\text{using a loop (see statics supplement or student ed. of Matlab)}
\end{align*}
\]
for i=1:n-1
x(i)=(v(i+1)+v(i))*(0.1)/2
end
for i=2:n
a(i)=(v(i-1)-v(i))/0.1
end
%to see the result plotted use the following
plot(t,x), xlabel ('t*dt or elapsed time'), title ('[position vs time')
plot(t,a), xlabel ('t*dt or elapsed time'), title ('acceleration vs time')
Note these plots are not given but appear in the text.

1.24 Solution:

Here is the Mathcad solution. From the plot estimate the following data:

\[ x_0 = 0 \quad x_1 = 0 \quad x_2 = 0 \quad x_3 = .02 \quad x_4 = .04 \quad x_5 = 0 \]
\[ x_6 = .1 \quad x_7 = .18 \quad x_8 = .25 \quad x_9 = .38 \quad x_{10} = .5 \]
\[ x_{11} = 0 \]

These are measured every 0.1 sec. Thus the velocity becomes:

\[ v_n = \frac{x_{n+1} - x_n}{0.1} \]

Which is plotted below

Next estimate the acceleration from the velocity:

\[ a_n = \frac{v_{n+1} - v_n}{0.1} \]

which is plotted below
Next the equivalent Matlab code is given (without the plots as they look the same). This is the same type of code used in problems 1.11, 1.12 and 1.23.

\[ x = [0 \ 0 \ 0 \ 0.02 \ 0.04 \ 0.1 \ 0.18 \ 0.25 \ 0.38 \ 0.7]; \]
\[ t = 0:0.1:10; \ dt = 0.1; \]
\[ n = \text{length}(x); \]
\[ v = 0\times x; \ a = 0\times v; \]
\[ \text{for } i = 1:n-1 \]
\[ v(i) = (x(i+1)-x(i))/0.1 \]
\[ a(i) = (v(i+1)-v(i))/0.1 \]
\[ \text{end} \]
\[ \text{plot (t,v), title ('velocity versus time')} \]
\[ \text{plot (t,a), title ('acceleration versus time')} \]

1.25 This is of second order and is linear \( x(t) \). The term \( \sin(\pi t) \) is nonlinear but in \( t \), not \( x \).

1.26 This is first order and linear in \( v(t) \).

1.27 This is first order and linear in \( v(t) \)

1.28 This is second order in \( \theta \) and nonlinear because of the term \( \sin \theta(t) \).

1.29 This is second order in \( \theta(t) \) and nonlinear because of the terms \( (d\theta/dt)|d\theta/dt| \) and \( \sin \theta(t) \).

1.30 This is first order in \( v \) and nonlinear because of the term \( v^2 \).

1.31 This is second order and linear in \( x(t) \).

1.32 This is still linear of second order in \( x(t) \).
1.33 Solution:
\[ a(t) = 3t^2 - 4t, \quad x(0) = 0, \quad v(0) = 0 \]
\[ v(t) = \int_0^t (3t^2 - 4t) dt = t^3 - 2t^2 \text{ ft/sec} \]
\[ x(t) = \int_0^t (t^3 - 2t^2) dt = \frac{1}{4} t^4 - \frac{2}{3} t^3 \text{ ft} \]

The plot from Mathcad is:
\[ x(t) = 0.25 t^2 - \frac{2}{3} t^3 \quad \text{for} \quad t := 0, \, 0.01 \ldots 2 \]

![Plot](FIGURE S1.33)

1.34 Solution:
\[ a(t) = 40t \cos \pi t \]
\[ \int_0^t dv = \int_0^t 40x \cos \pi x dx \]
\[ v(t) = 3 + \frac{40}{\pi} [\cos \pi t + t\pi \sin \pi t - 1] \]
\[ \int_0^t dx = 3t + \frac{40}{\pi^2} \int_0^t (\cos \pi x + x\pi \sin \pi x - 1) dx \]
\[ x(t) = 5 + 3t + \frac{40}{\pi^2} \sin \pi t - \frac{40}{\pi} t + \frac{40}{\pi^3} [\sin \pi x - \pi x \cos \pi x]_0^t \]
\[ x(t) = 5 + (3t - \frac{40}{\pi}) + \frac{40}{\pi^3} [2 \sin \pi t - \pi t \cos \pi x] \]
The following is typed in Mathcad to produce the desired plot

\[ t = 0, .01 .. 10 \]

\[ x(t) = 5 + \left( 3 - \frac{40}{\pi^2} \right) \cdot t + \frac{40}{\pi^3} \cdot (2 \cdot \sin(\pi \cdot t) - \pi \cdot t \cdot \cos(\pi \cdot t)) \]

The equivalent Matlab code is:

```
sys t
x=5+(3-40/pin2)*t+(40/pin3)*(2*sin(pi*t)-pi*t*cos(pi*t))
ezplot(x,[0,10])
```

(1.35) For the car \( t_0 = 0, \ x_0 = 0, \ v_f = 60 \text{ mph} = 88 \text{ ft/sec}, \ t_f = 6 \). For constant acceleration equation (1.28) yields \( at = v_f \) or \( a_{\text{car}} = \frac{88}{6} = 14.67 \text{ ft/sec}^2 \). In terms of g’s, \( a = \frac{14.67}{32.2} = 46\% g \). For the sprinter \( t_0 = 0, \ x_0 = 0, \ v_f = 10 \text{ m/s} \) and \( x_f = 15 \text{ m} \). From equation (1.30)

\[ a = \frac{v_f^2 - v_0^2}{2(x_f - x_0)} = \frac{10^2 - 0}{2(15 - 6)} = 3.33 \text{ m/s}^2 \]

In terms of g, \( a = \frac{3.33}{9.81} = 34\% g \).

(1.36) a) This is constant acceleration with \( a_0 = -9.81 \text{ m/s}^2 \) (taking positive \( x \) as up), \( v_0 = 10 \text{ m/s} \) and \( x_0 = 0 \). Equation (1.30) relates the displacement, acceleration and velocity. At the top of the motion \( v = 0 \), so eq. (1.30) becomes

\[ 0 = 2(-9.81)(x_{\text{top}}) + v_0^2 \text{ or } x_{\text{top}} = 5.1 \text{ m} \]
b) Equation (1.29) relates constant acceleration to velocity, time and position.
When the ball returns to its initial position \( x = 0 \) and equation (1.29) becomes
\[
0 = \frac{a_0 t^2}{2} + v_0 t + 0.
\]
One solution is \( t = 0 \) which is the initial state. If \( t \neq 0 \), the relationship becomes
\[
t = \frac{v_0}{2a_0} = 2.038 \sim 2 \text{ sec}
\]

1.37 For constant acceleration \( v = at + v_0 \) and \( v^2 - v_0^2 = 2ax \) so
\[
a = \frac{v^2 - v_0^2}{2x} = \frac{(5 \times 10^6)^2 - (10^4)^2}{(2)(2)} = 6.3 \times 10^{12} \text{ m/s}^2,
\]
\[
t = \frac{v - v_0}{a} = \frac{5 \times 10^6 - 1 \times 10^4}{6.3 \times 10^{12}} = 8.0 \times 10^{-7} \text{ s}.
\]

1.38 Here \( a \) is a function of \( x \), so consider the development of part 2, eq. (1.17) and (1.18)
\[
adx = v dv \text{ or } -kxdx = v dv \text{ or }
\]
\[
-\frac{kx^2}{2} |^x_{x_0} = \frac{v^2}{2} |^x_{x_0}
\]
\[
kx_0^2 = kx^2 \text{ or }
\]
\[
v^2 = kx_0^2 - kx^2 \text{ or }
\]
\[
v(x) = \sqrt{k(x_0^2 - x^2)}.
\]
The position time relationship can be found from (1.20):
\[
t = \int_0^t dt = \int_{x_0}^x \frac{dx}{v(x)} = \int_{x_0}^x \frac{dx}{\sqrt{k(x_0^2 - x^2)}} = \frac{1}{\sqrt{k}} \sin^{-1} \left( \frac{x}{x_0} \right) |^x_{x_0}.
\]
Rearranging and solving for \( x(t) \) yields
\[
t = \sqrt{\frac{1}{k}} \sin^{-1} \left( \frac{x}{x_0} \right) |^x_{x_0} \text{ or } \sqrt{kt} = \sin^{-1} \left( \frac{x}{x_0} \right) - \sin^{-1}(1) \text{ or } \sqrt{kt} + \frac{\pi}{2} = \sin^{-1} \left( \frac{x}{x_0} \right) \text{ or }
\]
\[
x(t) = x_0 \sin(\sqrt{kt} + \pi/2) = x_0 \cos(\sqrt{kt}).
\]

1.39 As the elevator starts from rest with constant acceleration to its operating speed
\[
v^2 = 2ax \text{ and } v = at \text{ or }
\]
\[
t = \frac{v}{a} = \frac{(3 \text{ m/s})/25 \text{ m/s}^2}{= 1.2 \text{ sec}}
\]
and travels a distance of
\[
x = \frac{v^2}{2a} = \frac{(3 \text{ m/s})^2}{2(2.5 \text{ m/s}^2)} = 1.8 \text{ m}.
\]
Which is also the time and distance required to stop the elevator. Hence 2 \times 1.2 \text{ s} = 2.4 \text{ s} and 2 \times 1.8 \text{ m} or 3.6 \text{ m} are used up in starting up and slowing down, the remaining distance 200 m - 3.6 m or 196.4 m is traveled at a constant velocity of 3 m/s so
\[
t = \frac{x}{v} = \frac{196.4}{3 \text{ m/s}} = 65.5 \text{ sec}.
\]
The total time is then
\[1.2 + 05.5 + 1.2 = 67.9 \text{ sec}\] or a little over one minute.

1.40 Since \( a = -c^2x \) case 2 applies and
\[
\frac{v^2 - v_0^2}{2} = \int_{x_0}^{x} -c^2 x \, dx = -c^2 \left( \frac{x^2 - x_0^2}{2} \right)
\]
or \( v^2 = v_0^2 + c^2 x_0^2 - c^2 x^2 \). Substitution of \( x_0 = 0, x = 10, v_0 = 30, v = 0 \), yields \( 0 = 30^2 + 0 - c^2 10^2 \) or \( c = 3 \).

1.41 Given \( a(x) = -cx^2 \), \( x_0 = 0 \), \( t_0 = 0 \), \( v(0) = v_0 \) we want to determine \( v(x) \). From eq. (1.17)
\[-c \int_{x_0}^{x} x^2 \, dx = \int_{v_0}^{v} v \, dv = \frac{1}{2}(v^2 - v_0^2)\]
or \( \frac{1}{2}(v^2 - v_0^2) = -\frac{c}{3} x^3 \) or \( v(x) = \sqrt{v_0^2 - \frac{2c}{3} x^3} \)

1.42 Since \( a \) is given as a function of velocity, case 3 applies: \( dx = v \, dv / f(v) \)
upon integrating
\[x - x_0 = \int_{v_0}^{v} v \, dv / (-v) = -v + v_0.\]
Since \( x_0 = 0 \) and \( v = 0 \) when it comes to rest, \( x = 750 \text{ mm} \).

1.43 From the problem statement \( y_0 = 40 \text{ km} \), \( v_0 = 6000 \text{ km/s} \) calculate an expression for \( y \). Here acceleration is a function of position, so equations (1.17)-(1.20) apply. Given
\[a(y) = -g_0 \frac{R^2}{(R+y)^2}, \quad g_0 = 9.81 \text{ m/s}^2, \quad R = 6370 \times 10^3.\]
At \( t = 0 \), \( y_0 = y(0) = 40 \times 10^3 \text{ m} \), \( v_0 = v(0) = 6000 \text{ m/s} \)
Note \( y_{\text{max}} \) will occur when \( v = 0 \). So compute \( v \).
\[a = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = -g_0 \frac{R^2}{(R+y)^2}.\]
Integrating yields
\[
\int_{v_0}^{0} v \, dv = -g_0 R^2 \int_{y_0}^{y_m} \frac{dy}{(R+y)^2} = g_0 R^2 \left[ \frac{1}{R+y_m} - \frac{1}{R+y_0} \right].
\]
Thus
\[
\frac{g_0 v^2}{2} - v_0^2 = g_0 R^2 \left[ \frac{1}{R+y_m} - \frac{1}{R+y_0} \right].
\]
Thus
\[
\frac{6000^2}{2} = g_0 R^2 \left[ \frac{1}{R+y_0} - \frac{1}{R+y_m} \right] \quad \text{or} \quad 4.552 \times 10^{-8} = \left[ \frac{1}{R+y_0} - \frac{1}{R+y_m} \right].
\]
Solving for \( y_m \) yields \( y_m = 2575 \).
1.44 Solution:
\[ \int_{v_{esc}}^{0} v^2 dv = -\frac{g_0 R^2}{(R+y)^2}\int_{y_0}^{\infty} \frac{dy}{(R+y)^2} \]
\[ \frac{v^2}{2} = -2g_0 R^2 \int_{y_0}^{\infty} \frac{dy}{(R+y)^2} \]
\[ -v_{esp}^2 = -2g_0 R^2 \int_{y_0}^{\infty} \frac{dy}{(R+y)^2} = g_0 R^2 \]
\[ \lim_{y_m \to \infty} \left[ \frac{1}{R+y_m} - \frac{1}{R+y_0} \right] = -2g_0 R \frac{1}{(R+y_0)}. \]
That is \(-v_{esp}^2 = -2g_0 R^2 \frac{1}{(R+y_0)} \)
Then: \(v_{esp} = R \sqrt{\frac{2g_0}{(R+y_0)}} = 11.14 \text{ km/s} = 11.14 \times 10^3 \text{ m/s} \)

1.45 Given: \(a(v) = -cv = -0.4v, \ v_0 = 100 \text{ km/hr.} \)
Since \(a = dv/dt\) we have
\[ \int_{v_0}^{v} \frac{dv}{v} = -ct \text{ or } \ln \frac{v}{v_0} = -ct. \]
Thus \(v = v_0 e^{-ct}. \)
But \(v = dx/dt\) so that
\[ x = x_0 + v_0 \int_{0}^{t} e^{-ct} dt = x_0 + \left( -\frac{v_0}{c} \right) e^{-ct}\bigg|_{0}^{t} \]
Thus \(x(t) = x_0 + \frac{v_0}{c}(1 - e^{-ct}). \)
With \(x_0 = 0, \ v_0 = 100\) and \(c = 0.4\) this becomes \(x(t) = 250(1 - e^{-0.4t}). \)

1.46 Solution:
\(a(t) = 5\sin(20t) \text{ m/s}^2 \ x(0) = 1 \text{ m} \) and \(v(0) = 3 \text{ m/s}. \)
Integrating:
\[ \int_{0}^{v_0} dv = v - v_0 = \int_{0}^{t} 5\sin 20\alpha d\alpha = -\frac{5}{20}(\cos 20t - 1) \]
where \(\alpha\) is used as the “dummy” variable of integration. Then
\[ v(t) = 3 - 0.25 \cos 20t + 0.25 = 3.25 - 0.25 \cos 20t \text{ m/s.} \]
Integrating again yields
\[ x(t) = x_0 + 3.25t - 1.25 \times 10^2 \sin 20t \]
\[ x(t) = 1 + 3.25t - 0.0125 \sin 20t \text{ m} \]
1.47 Solution:

\[ v_0 = 0.6 \text{ ft/s}, \quad a = -v^3 \text{ ft/s}^2 \]

Thus this is case 3 on page 19. However a straightforward integration of
\[ a = \frac{dv}{dt} \]

Then \[ dt = -\frac{dv}{v^3} \] and integrating yields
\[ t = -\frac{1}{2v^2} + 1.389. \]

Rearrange to get
\[ v(t) = \left(\frac{1}{2t + 2.778}\right)^{1/2} \text{ ft/s}. \]

At \( t = 4 \) sec, \( v(4) = 0.305 \text{ ft/s} \).

1.48 Follow example 1, because the acceleration is a function of velocity so case 3 is
used.

Note that \( a = \frac{dv}{dt} = g - cv^2 \) or \( \frac{dv}{g - cv^2} = dt \). Integrating both sides using the
stated initial conditions yields
\[ t = \int_0^v \frac{dv}{g - cv^2} = \frac{1}{c} \int_0^v \frac{dv}{\frac{g}{c} - v^2} = \left(\frac{1}{c}\right) \left(\frac{1}{2\sqrt{g/c}}\right) \ln\frac{\sqrt{g/c + v}}{\sqrt{g/c - v}} \]

for \( \frac{g}{c} > v^2 \) and \( 4\left(\frac{c}{g}\right)\frac{1}{2\sqrt{g/c}} \ln \left(\frac{v - \sqrt{g/c}}{v + \sqrt{g/c}}\right) \) for \( v^2 > g/c \), from using a table of
integrals. Thus there are two possibilities. For \( g/c > v^2 \);
\[ t = \frac{1}{2\sqrt{g/c}} \ln \frac{\sqrt{g/c - v}}{\sqrt{g/c + v}} \]

and solving for \( v(t) \) yields
\[ v(t) = \sqrt{\frac{2}{c}} \left(\frac{c\sqrt{gct} - 1}{c\sqrt{gct} + 1}\right); \quad g/c > v^2 \]

Note from this second expression the \( v^2 = g/c \) results in the expression \(-e^{2\sqrt{2ct}} = e^{2\sqrt{2ct}}\) which has no solution. Thus \( v \) cannot reach the value \( g/c \), i.e., \( v = g/c \) is
the driver’s terminal velocity. Next consider integrating again to calculate \( x(t) \), i.e., \( dx = vdt \) or
\[ \int_0^x dx = \sqrt{\frac{g}{c}} \int_0^t \left(\frac{e^{2\sqrt{gct}} - 1}{e^{2\sqrt{gct}} + 1}\right) dt \]

\[ = \sqrt{\frac{g}{c}} \left[ \int_0^t \frac{e^{2\sqrt{gct}}}{e^{2\sqrt{gct}} + 1} dt - \int_0^t \frac{dt}{e^{2\sqrt{gct}} + 1} \right] \]

\[ x(t) = \frac{1}{c} \left[ \ln \left(\frac{e^{2\sqrt{gct}} + 1}{e^{2\sqrt{gct}} + 1}\right) + \sqrt{gct - 0.693}\right] \]
1.49 This is a free fall problem or uniformly accelerated motion, where the acceleration is given as $g = 32.2 \, \text{ft/s}^2$, and the time traveled can be determined by equation (1.29) with $t_f$ as the given. Equation (1.29) becomes
$$x(t_f) = 30 \, \text{ft} = \frac{gt_f^2}{2} + v_0 t_f + x_0$$
Here $v_0 = 0$, since the ball is dropped, $x_0 = 0$ taking the window as the starting position and hence
$$\frac{gt_f^2}{2} = 30 \text{ or } t_f = \sqrt{\frac{30}{32.2}} = 1.365 \, \text{sec},$$
which is the time required to hit the ground. The expression for velocity under uniform or constant acceleration is equation (1.28) or (1.30). From (1.28)
$$v(t_f) = a_0(t_f) + v_0$$
$$v(1.365) = (32.2)(1.365) = 43.95 \, \text{ft/sec}.$$  

1.50 This is a case of uniform acceleration $a_0 = g = 32.2 \, \text{ft/s}^2$, with $v_0$ up, and $t_f = 1.71 \, \text{s}$. Using eq. (1.29) again with $v_0$ as the unknown yields
$$v(t_f) = 30 - \frac{(g)t_f^2}{2} + (-v_0)t_f + 0$$
Here $-v_0$ is used because $v_0$ is up and we have taken down as positive in writing a plus sign for $a_0 (= g)$. This is consistent with the solution to 1.49. Solving for $v_0$ yields
$$v_0 = \left[\frac{(g)t_f^2 - 30}{t_f}\right] / t_f = 9.987 \, \text{ft/sec}.$$  

1.51 Given $a_0 = 0.7g$ (constant acceleration), $v_f = 0$ (because the car comes to a stop). Convert mph to ft/s (60 mph = 88 ft/s, 45 mph = 66 ft/s, 30 mph = 44 ft/s) and use eq. (1.30)
$$v_f^2 = 2a_0(x_f - x_0) + v_0^2$$
where $v_f = 0$, $a_0 = -0.7 \, \text{g}$ (minus because it decelerates), $x_0 = 0$ (we start our distance measurement $t = 0$) then
$$x_f = \frac{v_0^2}{-2a_0} = \frac{v_0^2}{0.4g} = \frac{v_0^2 \, (\text{ft/sec})^2}{(1.4)(32.2) \, \text{ft/sec}^2}$$
so that a) $x_f = 171.78 \, \text{ft}$, b) $x_f = 96.63 \, \text{ft}$, c) $x_f = 42.95 \, \text{ft}.$  

1.52 Following the solution to 1.52m with $a_0 = 0.4g$ yields
$$x_p = \frac{v_0^2}{(-2)(-0.4)(32.2)}$$
so that a) $x_f = 300.6 \, \text{ft}$, b) $x_f = 169.1 \, \text{ft}$, c) $x_f = 75.2 \, \text{ft}.$
1.53 This is a uniform acceleration problem with $a_0 = 0.6g$. Since the car starts from rest $v_0 = 0$ and assume $x_0 = 0$. Let $v_f = 200$ mph $= 293.33$ ft/sec. Then the time to reach $v_f$ can be found from eq. (1.28)

$$v_f = a_0 t_f + v_0 \text{ or } t_f = \frac{v_f - v_0}{a_0} = \frac{293.33}{0.6} = 15.183 \text{ s.}$$

The distance traveled is found from eq. (1.29) to be

$$x_f = \frac{a_0 t_f^2}{2} = \frac{(0.6)(32.2)(15.183)^2}{2} = 2226.9 \text{ ft} = 0.422 \text{ mile}\n$$

1.54 Both cars undergo uniform acceleration $a_A = 0.9g$ and $a_B = 0.85g$. Let them start at $t = 0$ in the same place from rest, i.e., $x_a(0) = x_B(0) = v_A(0) = v_B(0) = 0$. Car A travels 1,000 m or takes the time determined by equation (1.29)

$$x_A(t_f) = 1000 = \frac{(0.9g)(t_f^2)}{2}$$

Then $t_f = 15.1$ sec. During this time car B travels a distance determined by

$$x_B(15.1) = \frac{a_B(t_f)^2}{2} = \frac{(0.85)(0.81)(15.1)^2}{2} = 950.6 \text{ m}$$

So car A is $1000 - 950.6 = 49.4 \text{ m}$ ahead of car B when it crosses the finish line. Note that if $t_f = 15.05$ s is used and not rounded off, then the distance becomes $55.47 \text{ mm}$ instead of $49.4 \text{ m}$.

1.55 This can be solved several ways including graphically by computing the area under the acceleration curve to generate the velocity, and the area under the velocity versus time curve to compute the position:

First write the acceleration during each interval. For $0 < t < 50s$, $a(t) = 2 \text{ m/s/s}$.

For $50 < t < 70s : a(t) = 0$, for $70 < t < 100, a(t) = 15(t - 70)$. Last for $100 > t > a(t) = 0$.

Now calculate the area under the curve in each of these intervals being careful to use the appropriate initial conditions at the beginning of each interval:

$$0 < t < 50 \quad v(t) = 20t \text{ m/s}$$

$$50 < t < 70 \quad v(t) = 1000 \text{ m/s}$$

$$70 < t < 100 \quad v(t) = 7.5(t - 70)^2 + 1000 \text{ m/s}$$

$$100 > t \quad v(t) = 7750 \text{ m/s} \text{ so that } v(120) = 7750 \text{ m/s}$$

Integrating each of these in the interval yields

$$0 < t < 50 \quad x(t) = 10t^2$$

$$50 < t < 70 \quad x(t) = 25000 + 1000(t - 50)$$

$$70 < t < 100 \quad x(t) = 2.5(t - 70)^3 + 1000(t - 70) + 45,000$$

$$t > 100 \quad x(t) = 7750(t - 100) + 142,500$$
This last expression yields
\[ x(120\text{s}) = 297,500 \text{ m} = 297.5 \text{ km}. \]

1.56 The equation to be solved is of the form
\[ \frac{dv}{dt} + v = e^t, \ v(0) = 0, \ x(0) = 1 \text{m} \]
Comparing with equation (1.32) identifies \( p(t) = 1 \) and \( f(t) = e^t \) so that the integrating factor becomes \( \lambda(t) = e^{\int dt} = e^t \).
According to equation (1.34) the solution is then
\[ v(t) = e^{-t}(\int e^t dt + C) = e^{-t}(\frac{1}{2}e^{2t} + C) \]
at \( t = 0, \ v(0) = 0 \text{ ft/s} \) so that \( C = 0.5 \) and \( v(t) = \frac{1}{2}(e^t - e^{-t}) \text{ m/sec} \)
= \( \sin h(t) \text{ m/sec} \)
Integrating again yields the displacement
\[ x(t) = x_0 + \int_0^t (e^\tau - 0.5)e^{\tau} d\tau \text{ when } x_0 = 1 \text{ m}. \] Thus \( x(t) = \frac{1}{2}(e^t + e^{-t}) \text{m} \)

1.57 The equation to be solved is of the form
\[ \frac{dv}{dt} + v = t, \ v(0) = 0, \ x(0) = 1, \ v(0) = 0 \]
Comparing to equation (1.32): \( p(t) = 1 \) and \( f(t) = t \). Thus the integrating factor becomes \( \lambda(t) = e^{\int dt} = e^t \)
According to equation (1.34) the solution becomes
\[ v(t) = e^{-t}(\int e^t dt + C) = e^{-t}[t - 1 + C] = t - 1 + Ce^{-t} \]
At \( t = 0, \ v(0) = 0 \text{ so that } 0 = -1 + C \text{ or } C = 1. \] Thus: \( v(t) = t - 1 + e^{-t} \text{ m/s} \)
Integrating again yields \( x_0 = 1 \text{m} \)
\[ x(t) = x_0 + \int_0^t (t - 1 + e^{-t}) dt = 1 + \frac{t^2}{2} - e^{-t} + 1, \text{ so that } x(t) = 2 - t + \frac{t^2}{2} - e^{-t} \text{m} \]

1.58 a) The equation to be solved is of the form
\[ \frac{dv}{dt} + tv = e^{-t^2/2} \]
Comparing this form to equation (1.32), identifies \( p(t) = t \) and \( f(t) = e^{-t^2/2} \).
Thus the integrating factor becomes
\[ \lambda(t) = e^{\int dt} = e^{t^2/2} \]
Next, equation (1.34) yields that the solution is
\[ v(t) = e^{-t^2/2}(\int e^{t^2/2}e^{-t^2/2} dt + c) = e^{-t^2/2}t + ce^{-t^2/2} \]
At \( 0, \ v(0) = 10 \text{ ft/s} \) so that \( c = 10 \) and \( v(t) = 10e^{-t^2/2} + e^{-t^2/2}t \)
Integrating again \(x(0) = 0\) yields \(x(t) = \int_0^t (10 + \tau)e^{-\tau^2/2}d\tau\) which yields the error function when integrated, i.e., \(x(t) = 12.5 \text{erf}(0.71t) - e^{t^2/2} + 1\).

b) This does not have an integrating factor, or other closed formed solution, so the solution must be found numerically by writing the equation in first order for and applying an Euler or Runge-Kutta solution. A Mathcad solution is shown.

\[
\begin{align*}
\Delta t &= 0.001 & t_i &= i \cdot \Delta t \\
x_0 &= 1 & v_0 &= 5 \\
a(v, t) &= -t^2 \cdot v - 1 - 3 \cdot t \\
\begin{bmatrix}
x_{i+1} \\
v_{i+1}
\end{bmatrix} &=
\begin{bmatrix}
x_i - v_i \cdot \Delta t \\
v_i - a(v_i, t_i) \cdot \Delta t
\end{bmatrix}
\end{align*}
\]

To solve this problem with Matlab, create and run the following code:

```matlab
x(1)=1; v(1)=5; t(1)=0;
dt=0.001;
for n=1:4000;
    x(n+1)=x(n)+v(n)*dt;
    v(n+1)=(-t(n).^2.*v(n)+1+3*t(n))*dt+v(n);
    t(n+1)=t(n)+dt;
end
plot(t,x),plot(t,v)
```

FIGURE S1.58
1.59 First solve the homogeneous equation $\ddot{x} + 5\dot{x} + 4x = 0$ by following eq. (1.41), assume a solution of the form $x(t) = Ae^{\lambda t}$ where $\lambda$ must satisfy

$$\lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1) = 0$$

so that $\lambda_{1,2} = -4, -1$. Thus the homogeneous solution is of the form $x_h(t) = A_1e^{-4t} + A_2e^{-t}$. The particular solution is guessed to be $x_p = a + bt$, of the form of the forcing function where $a$ and $b$ are to be determined. Substitution of the assumed form for $x_p(t)$ into the equation of motion yields

$$5b + 4(a + bt) = 3t + 0t^0$$

Comparing coefficients of $t$ and $t^0$ yields

$$5b + 4a = 0 \text{ and } 4b = 3$$

so that $b = 3/4$ and $a = -\frac{5}{4}(\frac{3}{4}) = -\frac{15}{16}$. Thus $x_p = -\frac{15}{16} + \frac{3}{4}t$. This is called the method of undetermined coefficients. The total solution is the sum $(x = x_h + x_p)$ so that

$$x(t) = A_1e^{-4t} + A_2e^{-t} + \frac{3}{4}t - \frac{15}{16}$$

To determine the coefficients $A_1$ and $A_2$ apply the initial conditions

$$x(0) = 0.5 = A_1 + A_2 - \frac{15}{16}$$
$$v(0) = 0 = -4A_1 - A_2 + \frac{3}{4}$$

which represents two equations in the two unknowns $A_1$ and $A_2$. Solving yields $A_1 = -0.229$ and $A_2 = 1.667$ and hence: $x(t) = -0.229e^{-4t} + 1.667e^{-t} + \frac{3}{4}t - \frac{15}{16}$

1.60 Define $\omega^2 = \frac{k}{m} = \frac{4}{1} = 4$ and $\zeta = \frac{c}{2m\omega} = 2.795 > 0$ so the system is over damped. Then the problem in standard form is

$$\ddot{x} + 2\zeta \omega \dot{x} + \omega^2 x = \ddot{x} + 5\dot{x} + 4x = 0$$

Assume solutions of the form $x = Ae^{\lambda t}$. The characteristic equation becomes

$$\lambda^2 + 5\lambda + 4 = 0$$

which has roots $\lambda_1 = -1, \lambda_2 = -4$. Thus the general solution is of the form

$$x(t) = A_1e^{-t} + A_2e^{-4t}$$

Applying the initial condition yields

$$x(0) = 5 = A_1 + A_2$$
$$v(0) = 0 = -A_1 - 4A_2$$

which is a system of two linear equations in the two unknowns $A_1$ and $A_2$. Solving yields $A_1 = \frac{20}{3}$ and $A_2 = -\frac{5}{3}$. Thus the solution is $x(t) = \frac{20}{3}e^{-t} - \frac{5}{3}e^{-4t}$ and $v(t) = \frac{20}{3}(-e^{-t} + e^{-4t})$.
The equation of motion has the form (after dividing by 1000)

\[ a = \frac{dv}{dt} = -cv - 400x, \ v(0) = 0, \text{ and } x(0) = 0.01 \text{m} \]

Following along with equation (1.51), the Euler method of integration yields

\[
\begin{bmatrix}
v_{i+1} \\
x_{i+1}
\end{bmatrix} = \begin{bmatrix}
v_i - cv_i\Delta t - 400x_i\Delta t \\
x_i + v_i\Delta t
\end{bmatrix}, \begin{bmatrix}
v_0 \\
x_0
\end{bmatrix} = \begin{bmatrix}
0 \\
0.01
\end{bmatrix}
\]

Using a high level language (Matlab, Mathcad or Mathematica) yields (some students may know the analytical solution for this equation. Others will know how to use the more sophisticated higher-order Runge-Kutta integration the following plot. Values of \( c \) are varied until the plot produces only two oscillations.)

\[
\Delta t := 0.01 \quad c = 15
\]

\[
i := 0.. \frac{1.5}{\Delta t}
\]

\[
\begin{bmatrix}
v_0 \\
x_0
\end{bmatrix} = \begin{bmatrix}
0 \\
0.01
\end{bmatrix}
\]

\[
\begin{bmatrix}
v_i + 1 \\
x_i + 1
\end{bmatrix} = \begin{bmatrix}
v_i - c\cdot v_i \cdot \Delta t - 400\cdot x_i \cdot \Delta t \\
x_i - v_i \cdot \Delta t
\end{bmatrix}
\]

Note here that the oscillation dies out at about \( t = 1 \) second, for a value of \( c = 15 \), or a damping value of 15,000 kg/s.

The Matlab code for doing this is given below using an Euler method. This can also be done using ODE which involves a Runge-Kutta routine. Create the

**FIGURE S1.61**
following Matlab code then run it with different values of c until the desired response results:

c=15
x(1)=0.01;v(1)=0.0;t(1)=0;
dt=0.01;
for n=1:150;
    x(n+1)=x(n)+v(n)*dt;
    v(n+1)=v(n)-c*v(n)*dt-400*x(n)*dt
end
plot(t,x)

Run this Matlab code with various values of c until the response decays within two cycles as desired.

1.62 Following the development of the numerical integration section equation (1.51) becomes

\[
\begin{bmatrix}
    v_{i+1} \\
    x_{i+1}
\end{bmatrix} = \begin{bmatrix}
    v_i - 900v_i \Delta t - 4000(x_i)^2 \Delta t \\
    x_i + v_i \Delta t
\end{bmatrix}
\]

with initial condition \( v_0 = 0 \) and \( x_0 = 20 \) mm. The Mathcad code is:

\[
\begin{align*}
i &:= 0..1000 \\
\Delta t &= .001 \\
\begin{bmatrix}
    v_0 \\
    x_0
\end{bmatrix} &= \begin{bmatrix}
    0 \\
    20
\end{bmatrix} \\
\begin{bmatrix}
    v_i \\
    x_i
\end{bmatrix} &= \begin{bmatrix}
    v_{i-1} - 900v_{i-1} \Delta t - 4000(x_{i-1})^2 \Delta t \\
    x_{i-1} + v_{i-1} \Delta t
\end{bmatrix}
\end{align*}
\]

FIGURE S1.62
The equivalent Matlab code can be either Euler (see previous problem) or Runge-Kutta Method. To use RK, first save the following Matlab code under Onept62.m:

```matlab
function xdot=onept62(t,x)
xdot=[x(2);-900*x(2)-4000*x(2)-4000*x(2)*x(2)];
% the equation of motion
Then the following commands will compute and plot the solution
EDU>tspan=[0 1] % defines the time interval of interest
EDU>x0=[20;0]; %enters the initial conditions, displacement first
EDU>ode45(*onept62',tspan,x0); % calls the RK routine and applies
% it to 1.62.

1.63 Following the development of the numerical integration section (1.51) becomes

\[
\begin{bmatrix}
v_{i+1} \\
x_{i+1}
\end{bmatrix} = \begin{bmatrix} v_i - 90v_i\Delta t - 100x_i^3\Delta t \\ x_i + v_i\Delta t \end{bmatrix}
\]

with initial condition \( v_0 = 0 \) and \( x_0 = 10 \) mm.

The Mathcad code is:

\[
\Delta t = 0.001 \quad N = 500 \quad i = 0..N \quad t_i = i:\Delta t \quad c = 90 \quad k = 100
\]

\[
v_0 = 0 \quad x_0 = 10 \quad a(v,x) = -c \cdot v - k \cdot x^3
\]

\[
\begin{bmatrix} x_{i+1} \\ v_{i+1}
\end{bmatrix} = \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - a(v_i,x_i) \cdot \Delta t \end{bmatrix}
\]

![FIGURE S1.63](image)

The catch gets near zero within \( \frac{1}{20} \) second.
The Matlab code is to prepare the following file named onept63.m:
Function xdot=onept63(t,x)
c=20;k=100;
xdot=[x(2);-c*x(2)-k*x(1)^3];
Then type the following in the command window:
EDU>tspan=[0 0.5];
EDU>x0=[10;0];
EDU>ode45('onept63',tspan,x0);

1.64 Following the solution to 1.63 equation (1.51) becomes:
\[
\begin{bmatrix}
    v_{i+1} \\
    x_{i+1}
\end{bmatrix}
= \begin{bmatrix}
    v_i - cv_i \Delta t - 100(x_i)^3 \Delta t \\
    x_i + v_i \Delta t
\end{bmatrix}
\]

Repeat the numerical solution to problem 1.63 with successively smaller values of damping \(c\) each time until the solution oscillates twice before coming to rest. A value of about \(c = 20\) 1/s comes close as illustrated.

The Mathcade code is:
\[
\Delta t = 0.001 \quad N = 500 \quad i = 0..N \quad t_i = i \cdot \Delta t \quad c = 20 \quad k = 100
\]
\[
v_0 = 0 \quad x_0 = 10 \quad a(v,x) = -c \cdot v - k \cdot x^3
\]
\[
\begin{bmatrix}
    x_{i-1} \\
    v_{i-1}
\end{bmatrix}
= \begin{bmatrix}
    x_i + v_i \cdot \Delta t \\
    v_i + a(v_i,x_i) \cdot \Delta t
\end{bmatrix}
\]

\[\text{FIGURE S1.64}\]
The Matlab code requires the following file saved as onep64.m:

function xdot=onept64(t,x)
c=20;k=100; xdot=[x(2);-c*x(2)-k*x(1)^3];

The type the following in the command window

EDU>tspan=[0 0.5];
EDU>x0=[10;0];
EDU>ode45('onept64',tspan,x0);

1.65 Solution: First set up the Euler form of the equation for numerical integration:

\[
\begin{bmatrix}
  v_{i+1} \\
  x_{i+1}
\end{bmatrix} = \begin{bmatrix}
  v_i - c v_i |v_i| \Delta t - 4k x_i |x_i| \Delta t \\
  x_i + v_i \Delta t
\end{bmatrix}
\]

Then resolve for various values of \( c, k \) and \( x_0 \) until a response that dies out in one oscillation results. There are many answers, the plot shows this is achieved for \( x_0 = 0.01 \) m, \( k = 400 \) 1/m\( \cdot \)s\( ^2 \), and \( c = 1000 \) m\( ^{-1} \). Another solution is \( x_0 = 2 \), \( c = 6 \) and \( k = 40 \).

The Mathcad solution is:

\[
\begin{align*}
  c &= 1000 \quad \Delta t = 0.01 \quad x_0 = 0.01 \\
  v_0 &= 0 \\
  i &= 0..3000 \\
  \begin{bmatrix}
    v_{i+1} \\
    x_{i+1}
  \end{bmatrix} &= \begin{bmatrix}
    v_i - c \cdot v_i \cdot |v_i| \cdot \Delta t - 400 \cdot (x_i \cdot |x_i|) \cdot \Delta t \\
    x_i + v_i \cdot \Delta t
  \end{bmatrix}
\end{align*}
\]

FIGURE S1.65
Save the following Matlab code as a file named “onept65.m”:

function xdot=onept65(t,x)
    c=1000; k=400;
    xdot=[x(2);-c*x(2)*abs(x(2))-k*x(1)*abs(x(1))]

In the command window:

EDU>tspan=[0 30];
EDU>x0=[0.01; 0];
EDU>ode45('onept65',tspan,x0);

1.66 Assuming \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) so that \( m\ddot{\mathbf{r}} = m\ddot{x} \mathbf{i} + m\ddot{y} \mathbf{j} + m\ddot{z} \mathbf{k} \). Then \( m\ddot{\mathbf{r}} = -g\mathbf{j} \) yields \( \ddot{x} = 0 \), \( \ddot{y} = -g/m \) and \( \dddot{z} = 0 \). These are linear, decoupled equations.

1.67 Yields the 3 scalar equations \( \ddot{x} + c\dot{x} = 0 \), \( \ddot{y} + c\dot{y} + g = 0 \) and \( \dddot{z} + c\dot{z} = 0 \) which are decoupled, linear equations.

1.68 Assuming \( \mathbf{r} = x \mathbf{i} + v_y \mathbf{j} + z \mathbf{k} \) and \( \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \) yields

\[
\ddot{x} = -c\sqrt{(v_x^2 + v_y^2 + v_z^2)} \, v_x, \quad \ddot{y} = -c\sqrt{v_x^2 + v_y^2 + v_z^2} \, (v_y), \quad \ddot{z} = -c\sqrt{v_x^2 + v_y^2 + v_z^2} \, (v_z).
\]

These are coupled, nonlinear equations.

1.69 Assuming \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) and \( \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \)

\[
\ddot{x} = 3t^2, \quad \ddot{y} = -\sin(\pi t) \quad \ddot{z} = xz
\]

The \( x \) and \( y \) equations are linear and decouple.

The \( z \) equation is nonlinear and coupled to \( x \).

1.70 Consider the plane trajectory equations given by eq. (1.74). In this case we know \( x_f = 450 \text{ ft}, z_f = 12 \text{ ft}, g = 32.2 \text{ ft/s}^2, x_0 = 0, z_0 = 0 \) and \( v_0 = 130 \text{ ft/s} \). Thus equation 1.73 becomes

\[
450 = (130 \cos \theta) t + 0 \\
12 = -16.2t^2 + (130 \sin \theta) t
\]

which is two nonlinear algebraic equations in two unknowns \( t \) and \( \theta \). Solving yields \( t = 6.86 \text{ sec}, \theta = 1.042 \text{ rad} \) (59.7°) and \( t = 4.088, \theta = 0.561 \text{ rad} \) (32.143°).
These solutions are found using Mathcad. Note that there are two solutions found by taking different initial guesses to the iterative solution of these non-linear algebraic equations. One solution has a low angle which would drive into the bunker and one (correct) that has a larger angle which will loft onto the green.

1.71 Choose \((0,0)\) in the \(x - z\) plane to be on the ground so that \(v_z(0) = 2\) m/s, \(z_f = 0\), \(x_0 = 0\), \(z_0 = 3\) m, \(x_f = d\), \(x_f = 0\). Then equation (1.73) becomes \(d = 2t\), \(0 = \frac{9.81}{2}t^2 + v_0(0) + 3\). Combining \(4.905t^2 = 3\) and \(t = \frac{d}{2}\) yields \(d^2 = \frac{12}{4.905}\) or \(d = 1.56\) m.

1.72 Looking at the top half spray, Eq. 1.73 becomes
\[
d_2 \cos 10^\circ = 20 \cos 70^\circ t + 0 \tag{1}
\]
\[
d_2 \sin 10^\circ = -16.1t^2 + 20 \sin 70^\circ t + 0 \tag{2}
\]
which is a system of two equations in the two unknowns: \(d_2\) and \(t\).

From (1) \(t = \frac{d_2 \cos 10^\circ}{20 \cos 70^\circ} = 0.1459d_2\) \(d_2 = \frac{20 \sin 20^\circ(0.1439) - \sin 10^\circ}{(16.1)(0.1439)^2}\) or \(d_2 = 7.588\) ft.

Next consider the spray to the left:
\[
d_1 \cos 10^\circ = 20 \cos 50^\circ t + 0 \tag{1}
\]
\[
0 = -16.1t^2 + 20 \sin 50^\circ t + d_1 \sin 10^\circ \tag{2}
\]
From (1) \(t = .0766d_1\) or \(t^2 = .005268d_1^2\) and eq. (2) becomes
\[
d_1 = \frac{(20)^2 \cos 50^\circ}{(16.1)(\cos^2 10^\circ)}(\cos 10^\circ \tan 50^\circ + \sin 10^\circ) = 14.26\) ft.

1.73 Working with equation 1.73 for projectile motion, let the hose be at \(x_0 = z_0 = 0\) and assume it hits at \(x(t_f) = x\) and \(z(t_f) = 0\), then eq. (1.73) becomes
\[
x = v_0 \cos \theta t_f\] and \(0 = g \frac{t_f^2}{2} + v_0 \sin \theta t_f\). Solving this last expression for \(t_f\) yields \(t_f = \frac{2v_0 \sin \theta}{g}\), the time to hit the ground. Then from the expression for \(x\)
\[
x(t_f) = \frac{2v_0^2}{g} \sin \theta \cos \theta
\]
The max value of \(x\) occurs at \(dx/d\theta = 0\) or \(\frac{2v_0^2}{g}(-\sin^2 \theta + \cos^2 \theta) = 0\). This requires \(\sin \theta = \cos \theta\) or \(\theta = 45^\circ\), the value at which \(x_f\) will be maximum.
1.74 From the problem statement, taking the batters “foot” as the origin, the value of \( x_0 = 0 \), \( z_0 = 4 \) ft, \( v_0 = 140 \) ft/s, \( \theta = 20^\circ \) and (since it hits the ground) \( z_f = 0 \). Equation (1.73) then becomes
\[
x(t_f) = 14 - \cos 20^\circ \ t_f \text{ and } 0 = -16.1t_f^2 + 140 \sin 20^\circ t_f + 4
\]
Solving the last expression for \( t_f \) yields \( t_f = 3.055 \) sec. and -0.081 sec. Obviously the physical value is \( t_f = 3.055 \), which from the first equation yields \( x_f = 140(\cos 20^\circ)(3.055) = 402 \) ft.

1.75 From the projectile equation for \( z \):
\[
z = -16.1t_f^2 + 140 \sin 20^\circ t_f + 4 \]
The maximum value of the parabolic trajectory would occur at \( t_f/2 \) except the value of \( t_f \) calculated in 1.74 assumes the trajectory is 4 ft off the ground. The equation for time of flight is \( -16.1t_f^2 + 140 \sin 20^\circ t_f = 0 \) or \( t_f = 2.97 \) sec, and \( t_f/2 = 1.487 \) sec. Then \( z_{\text{max}} = -16.1 \left( \frac{2.92}{2} \right)^2 + 140 \sin 20^\circ \left( \frac{2.92}{2} \right) + 4 = 39.6 \) ft.

1.76 Consider the projectile equation 1.73 and first solve for \( v_0 \) so the ball just clears the bottom window. Picking a coordinate system 1m off the ground yields
\[
x_0 = z_0 = 0, \ x_f = v_0 \cos 30^\circ \ t_f, \ z_f = 2m = -\frac{9.81}{2}t_f^2 + v_0 \sin \theta t_f
\]
where \( x_f = 6.5 \) m. This yields two equations in two unknowns:
\[
6.5 = v_0(0.886)t_f \text{ or } t_f = \frac{7.5}{v_0}
\]
Thus \( v_0 = 12.56 \) m/s. With \( y_f = 3 \) m, this becomes \( v_0 = 19.18 \) m/s so that he must kick through with a speed: \( 12.56 < v_0 < 19.18 \) m/s.

1.77 The initial velocity is given as \( v_0 = 10 \) m/s, \( x_0 = z_0 = 0 \), \( z_f = 2m \) (for smallest and 3m for largest). \( x_f = 6.5 \) m so the first equation of (1.73) becomes (let \( t \) denote the time at which the ball reaches the window).
\[
6.5 = 1 - \cos \theta t \text{ or } t = 0.65/ \cos \theta
\]
The second projectile equation yields
\[
2 = -\left( \frac{9.81}{2} \right) \left( \frac{0.65}{\cos \theta} \right)^2 + 10 \sin \theta \left( \frac{0.65}{\cos \theta} \right)
\]
which can be solved numerically for \( \theta = 40.865^\circ \). Changing the value of \( z_f = 3 \) m and repeating yields \( \theta = 55.58^\circ \). Thus he needs to kick through at an angle between \( 40.9^\circ < \theta < 55.6^\circ \) to make it through the window with an initial velocity of 10 m/s. Each of these two equations have 2 solutions so it will also make it in for \( 59.44^\circ < \theta < 66.28^\circ \) on the high lofty solution.
1.78 The given values are $z_0 = d/\sqrt{2}$, $v_0 = 25 \text{ m/s}$, $\theta = 0$. Equation 1.73 becomes

$$x_f = 25t = \frac{d}{\sqrt{2}} \quad \text{or} \quad t = \frac{d}{25\sqrt{2}}, \quad 0 = -\frac{9.81}{2}t^2 + \frac{d}{\sqrt{2}}, \quad \text{or} \quad d = \frac{\sqrt{2}}{2} 0.81 \left( \frac{d^2}{(625)(2)} \right), \quad \text{so} \quad d = 180.2 \text{ m.}$$

![Graph](image)

1.79 Sample 1.14 gives the equation for a particle in projectile motion with wind resistances. The equations are nonlinear and coupled and must be solved numerically. The initial conditions of $x(0) = 0$, $v_x(0) = 25$, $y(0) = 0$, $v_y(0) = 0$ will allow the solution computed numerically following sample 1.14. The trajectory can then be plotted along with a line at $45^\circ$ representing the hill. The intersection will yield the value of $d$. Since we do not know $d$, it is best to put the coordinate system at the end of the ski run and let $z$ (or $y$) evolve in the negative direction. Such a line passing through the origin has slope -1 and can be written as $d = -x$, or $d_i = -x_i$ in incremental form. The Mathcad code is

\begin{verbatim}
\begin{align*}
i &:= 0..1100 \quad \Delta t = 0.005 \quad c = 0.04 \quad g = 9.81 \\
vx_0 &:= 25 \quad x_0 = 0 \quad vy_0 = 0 \quad y_0 = 0 \\
\begin{bmatrix}v x_{i+1} \\ x_i + 1 \\ v y_{i+1} \\ y_i + 1 \end{bmatrix} &:= \begin{bmatrix}v x_i - c \cdot v x_i \cdot \Delta t \\ v x_i + v x_i \cdot \Delta t \\ v y_i - (g - c \cdot v y_i) \cdot \Delta t \\ v y_i + v y_i \cdot \Delta t \end{bmatrix} \\
d_i &:= -x_i
\end{align*}
\end{verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image}
\caption{FIGURE S1.79}
\end{figure}

Form the plot, then cross about 111 m out or $d = 111/\cos 45^\circ = 157 \text{ m}$ down the incline.
The Matlab code for solving and plotting is given next with the plots suppressed as they are the same as the above:

```matlab
function xdot=onept78 (t,x):
c=0.04; g=9.81;
xdot=[x(2);-(c*x(2);x(4);-g-c*x(3));
In the command window:
EDU>tspan=[0 140]
EDU>x0=[0;25;0;0];
EDU>[t,x]=ode45('onept78',tspan,x0);
EDU>d=-x(:,1);
EDU>plot(x(:,1),x(:,3),'t',x(:,1),d,'*')
```

1.80 This is just a repeat of the previous problem with a more accurate nonlinear damping term. The Mathcad code is:

\[
\begin{align*}
    i &= 0 \ldots 1000 \\
    \Delta t &= 0.005 \\
    c &= 0.002 \\
    g &= 9.81 \\
    v_{x0} &= 25 \\
    x_0 &= 0 \\
    v_{y0} &= 0 \\
    y_0 &= 0 \\

    \begin{bmatrix}
        v_{x_{i+1}} \\
        x_{i+1} \\
        v_{y_{i+1}} \\
        y_{i+1}
    \end{bmatrix}
    &=
    \begin{bmatrix}
        \frac{v_{x_{i}} - c \cdot \sqrt{v_{x_{i}}^2 + v_{y_{i}}^2}}{\Delta t} \\
        x_{i} + \frac{v_{x_{i}} \cdot \Delta t}{v_{y_{i}}} \\
        \frac{v_{y_{i}} - g - c \cdot \sqrt{v_{x_{i}}^2 + v_{y_{i}}^2}}{\Delta t} \\
        y_{i} + \frac{v_{y_{i}} \cdot \Delta t}{v_{y_{i}}} 
    \end{bmatrix}
    \end{align*}
\]

\[d_i = -x_i\]

![FIGURE S1.80]
In this case the skier makes it about 108 meters out or so which is 108 = d \cos(45^\circ) so that \(d = 152.7\) m.

The Matlab code is the same as the previous problem except the “x dot” becomes:

\[ \text{xdot} = [x(2); -c*x(2)*sqrt(x(2)^2 + x(4)^2); x(4); -g - c*x(2)*sqrt(x(2)^2 + x(4)^2)]; \]

1.81 Solution: 300 yards = 900 ft so that \(x_f = x(t_f) = 900\). Given that the ball is at zero to start with \(x_0 = y_0 = 0\), and hits the ground at \(y_f = 0\). With \(\theta = 90^\circ\) given, the trajectory equations are

\[
\begin{align*}
900 &= v_0 \cos 9^\circ t + 0 \\
0 &= -16.1t^2 + v_0 \sin 9^\circ t + 0
\end{align*}
\]

Solve (1) for \(t\) and (2) for \(v_0\) to get \(v_0 = 306\) ft/s (about 209 mph).

1.82 Solution: 200 mph = 293.3 ft/s. Here \(x_0 = y_0 = 0, \theta = 9^\circ\). Then eq. (1.71) becomes

\[
\begin{align*}
x &= (293.3) \cos 9^\circ t = 289.7t \\
0 &= -16.1t^2 + 45.8t
\end{align*}
\]

From (2) \(t = 2.849\) sec so from (1) \(x = 825.6\) ft
1.83 This follows directly from the solution of sample 1.14 with the following values and equations: \( x_0 = y_0 = 0 \), \( c = 0.05 \), \( v_{x0} = 293.3 \cos 9^\circ \) ft/sec and \( v_{y0} = 293.3 \sin 9^\circ \) ft/s.

\[
\begin{align*}
\text{i} &= 0..600 \quad \Delta t = 0.005 \quad c = 0.05 \quad g = 32.2 \quad v_{x0} = 293.3 \cdot \cos (9^\circ) \\
x_0 &= 0 \quad v_{y0} = 293.3 \cdot \sin (9^\circ) \quad y_0 = 0
\end{align*}
\]

\[
\begin{bmatrix}
vx_i - 1 \\
x_i - 1 \\
vy_i - 1 \\
y_i - 1
\end{bmatrix} =
\begin{bmatrix}
vx_i - c \cdot vx_i \cdot \Delta t \\
x_i - vx_i \cdot \Delta t \\
vy_i - (g + c \cdot vy_i) \cdot \Delta t \\
y_i - vy_i \cdot \Delta t
\end{bmatrix}
\]

From the figure \( x = 758 \) ft (found by using the trace function in Mathcad). The Matlab code is given in the Matlab supplement.

1.84 Again use the projectile equations of eq. (1.73). Here: \( x_0 = 0 \), \( y_0 = 0 \) (so \( y_f = 3 \) ft), \( x_f = 20 \) ft, \( \theta = 45^\circ \) and hence

\[
20 = v_0 \frac{1}{\sqrt{2}} t \quad \text{or} \quad t = \frac{20 \sqrt{2}}{v_0}
\]

\[
3 = -16.1 t^2 + v_0 \cdot \frac{1}{\sqrt{2}} t
\]

Solving yields \( v_0 = \sqrt{\frac{(16.1)800}{17}} = v_0 = 27.53 \) ft/s.
1.85 Using the projectile motion equations with \( x_0 = y_0, x_f = 20 \text{ ft}, y_f = 3 \text{ ft} \) and \( v_0 = 30 \text{ ft/sec} \), equations (1.73) becomes

\[
\begin{align*}
20 &= (30) \cos \theta t \\
3 &= -16.1t^2 + (30) \sin \theta t
\end{align*}
\]

Substitution of \( t = \frac{2}{3 \cos \theta} \) from (1) into (2) yields the transcendental equation

\[
3 = -16.1 \left( \frac{4}{9 \cos^2 \theta} \right) + 20 \tan \theta
\]

Solving for \( \theta \) yields \( \theta = 33.7^\circ \) and \( 64.8^\circ \). Either angle will “work”, however the lower angle gives a trajectory up through the bottom of basket whereas the \( 64.8^\circ \) solution gives the lofty shot and then goes through the top of the hoop. The following Mathcad code solves the problem:

---

Specify the known parameters.
\[
\begin{align*}
v_0 &= 30 \\
x_0 &= 0 \\
y_0 &= 7 \\
x_f &= 20 \\
y_f &= 10 \\
g &= 32.2
\end{align*}
\]

Initial guess for time and angle
\[
\theta = 30 \text{- deg} \quad t_f = 3
\]

Given
\[
\begin{align*}
x_f &= v_0 t_f \cos(\theta) + x_0 \\
y_f &= -g \frac{t_f^2}{2} + v_0 t_f \sin(\theta) + y_0
\end{align*}
\]

Find \( \{ \theta, t_f \} \)
\[
\begin{bmatrix}
1.132 \\
1.568
\end{bmatrix}
\]
\[
\frac{1.132}{\text{deg}} = 64.859 \quad \text{Angle in degrees}
\]

A second solution can be found with a lower angle.
\[
\begin{align*}
v_0 &= 30 \\
x_0 &= 0 \\
y_0 &= 7 \\
x_f &= 20 \\
y_f &= 10 \\
g &= 32.2
\end{align*}
\]

Initial guess for time and angle These values are assumed less.
\[
\theta = 10 \text{- deg} \quad t_f = 1
\]

Given
\[
\begin{align*}
x_f &= v_0 t_f \cos(\theta) + x_0 \\
y_f &= -g \frac{t_f^2}{2} + v_0 t_f \sin(\theta) + y_0
\end{align*}
\]

Find \( \{ \theta, t_f \} \)
\[
\begin{bmatrix}
0.588 \\
0.801
\end{bmatrix}
\]
\[
\frac{0.588}{\text{deg}} = 33.69 \quad \text{Angle in degrees}
\]

The second solution is not valid as the ball would hit the net from below.
1.86 Using the projectile motion equations with \( x_0 = 0, y_0 = 10, \theta = 0, v_0 = 120 \text{ mph} = 176 \text{ ft/s} \) and \( y_f = 0 \) (i.e., hits the ground) yields

\[
x_f = 176t
\]  
(1)

\[
0 = -16.1t^2 + 10
\]  
(2)

From (2) \( t = 0.7885 \text{ sec} \) and from (1) \( x_f = 138.7 \text{ ft} \).

1.87 This again uses the projectile motion equation. a) Let \( x_0 = 0, x_f = 6 \text{ ft}, y_0 = 0, \) so \( y_f = 15 - 4 = 11 \text{ ft}, \theta = 80^\circ \) and the unknown is \( v_0 \). Equation (1.73) becomes

\[
x_f = 6 = (v_0 \cos 80^\circ)t + 0
\]  
(1)

\[
y_f = 11 = (v_0 \sin 80^\circ)t - 16.1t^2 + 0
\]  
(2)

Substitute \( t = \frac{6}{v_0 \cos 80^\circ} \) from (1) into (2) to get

\[
11 = 6 \tan 80^\circ - 16.1\left(\frac{6}{v_0 \cos 80^\circ}\right)^2
\]  
(3)

Solving yields \( v_0 = 28.9 \text{ f/s} \). b) Repeating (a) with \( x_f = 16 \) eq. (3) becomes

\[
11 = 16 \tan 80^\circ - 16.1\left(\frac{16}{v_0 \cos 80^\circ}\right)^2
\]

or \( v_0 = 41.4 \text{ ft/s} \).

1.88 This is circular motion with \( R = 150 \text{ ft}, a_t = 12 \text{ ft/s}^2 \). Compute the time \( t \) at which \( a_n = 24 \text{ ft/s}^2 \). From Eq. (1.84), \( a_t = \alpha r \) so that \( \alpha = \frac{a_t}{r} = \frac{12}{150} \text{ rad/s}^2 = 0.08 \text{ rad/s}^2 \) a constant. \( \frac{dw}{dt} = 0.08 \) so that \( w - w_0 = 0.08t \) or \( w = 0.08t + w_0 \). From (1.84) \( a_n = rw^2 = 25 = 150(w_0 + 0.08t)^2(1), \) where \( t_s \) is time to slip. Also at \( t_s, a = \sqrt{a_t^2 + a_r^2} = \sqrt{25^2 + 12^2} = 27.73 \text{ ft/s}^2 \) at slip. Solving (1), with \( \omega_0 = 0 \) for \( t_s \) yields \( t_s = \frac{1}{0.08} \left(\frac{25}{150}\right)^{1/2} = 5.104 \text{ s} \).

1.89 This is a circular motion with \( r = 2 \text{ m} \) and \( a_t(t) = 6 \sin \pi t \text{(m/s}^2\text{). The particle starts at rest so that } \theta(0) = \omega(0) = 0. \) For circular motion \( v(t) = \int_0^t a_t dt = \int_0^t 6 \sin \pi t = \frac{6}{\pi}(1 - \cos \pi t). \) Also \( a_r = \frac{v^2}{r} = \frac{1}{2}\left(\frac{6}{\pi^2}\right)(1 - \cos \pi t)^2 = \frac{12}{\pi^2}(1 - \cos \pi t)^2. \) From equation (1.81) taking the magnitude of \( a(t) \) yields

\[
a(t) = \sqrt{a_t^2 + a_r^2} = \sqrt{6^2 \sin^2 \pi t + \left[\frac{12}{\pi^2}(1 - \cos \pi t)^2\right]^2}.
\]
The plots follow:

\[ t = 0, 0.001, 2 \]
\[ v(t) = \frac{6}{\pi} \cdot (1 - \cos(\pi \cdot t)) \quad \text{at}(t) = 6 \cdot \sin(\pi \cdot t) \quad \text{an}(t) = \frac{18}{\pi^2} \cdot (1 - \cos(\pi \cdot t))^2 \]
\[ a(t) := \sqrt{\text{at}(t)^2 + \text{an}(t)^2} \]

**FIGURE S1.89**

The Matlab code for producing the plots is given in the following file:
```
syms t % declares t symbolic
v=(6/pi)*(1-cos(pi*t)); an=0.5*v^2;
at=6*sin(pi*t); a=sqrt(at^2+an^2)
ezplot(a,[0,2]), ezplot(v,[0,2])
```

1.90 From the solution to 1.89 \(a(t) = (36 \sin^3 \pi t + \frac{18^2}{\pi^4} (\cos \pi t - 1)^4)^{1/2}\). Find \(t_s\) when \(a(t) = 5\). The answer can be seen from the plot given in figure S1.89 or from solving
\[ 5^2 = 36 \sin^2 \pi t_s + \frac{18^2}{\pi^4} (\cos \pi t_s - 1)^4 \]
for \(t_s\) which has 2 solutions in the interval of interest. From Mathcad they are: \(t_s = 0.312\) s, and 1.688s.

1.91 Given \(r = 200\) m, \(v = 30\) km/hrs = \(\frac{30}{3600} \times \frac{10^3}{1} \text{ m/hr} = 8.33\) m/s. For circular motion, equation (1.81) yields
\[ a_n = \frac{v^2}{r} = \frac{8.33^2}{200} = 0.3469 \text{ m/s}^2 \text{ or } a_n = 0.35 \text{ m/s}^2. \]
1.92 This is circular motion starting from rest so that $\theta(0) = \omega(0) = 0$, with $r = 4$ m.

a) $\alpha(t) = 2t^2 \text{ r/s}^2$ so that $\frac{d\omega}{dt} = 2t$ and $\omega - \omega_0 = \frac{2}{3t}$ or $w(t) = \frac{2}{3}t^3$. Thus from eq. (1.84) $a_n = r(\omega)^2 = 4 \cdot \frac{1}{9}t^6$ and $a_t = 4(2t^2) = 8t^2$. At $t = 2s$, $a_n(2) = 113.8$, $a_t(2) = 32$ so that $a(t) = \sqrt{1138^2 + 32^2} = 118.2 \text{ m/s}^2$.

b) From eq. (1.82) $\frac{d^2s}{dt^2} = ra = 8t^2$ so that $dv = 8t^2dt$ or $v = \frac{8}{3}t^3$. Thus $ds = \frac{8}{3}t^3dt$ and $s(2) = \frac{8}{3} \int_0^2 t^3dt$ and the total distance traveled is $s = 10.67$ m.

1.93 Solution: $\alpha(t) = 3t^2 - 2t \text{ rad/s}^2$ with $\omega_0 = \theta_0 = 0$. Integrating yields $\frac{d\omega}{dt} = 3t^2 - 2t$ or $\int_0^t d\omega = \int_0^t 3t^2 - 2t \omega(t) = t^3 - t^2 \text{ rad/s}$. Likewise $d\theta = (t^3 - t^2)dt$ so that $\theta(t) = \int_0^t (t^3 - t^2)dt = \frac{t^4}{4} - \frac{t^3}{3} \omega$ or $\theta(t) = \frac{4}{3}t^3 - \frac{1}{3}t^3 \text{ rad}$.

1.94 Solution: $\alpha(t) = t \cos(\pi t) \text{ rad/s}^2$, $\omega_0 = 2 \text{ rad/s}$, $\theta_0 = 30^\circ = \frac{\pi}{6}$ rad. Thus $d\omega(t) = t \cos(\pi t)dt$ or upon integrating $\omega - 2 = \int_0^t x \cos\pi xdx = \frac{1}{\pi}[\cos(\pi t) + \pi t \sin(\pi t)]$ so that $\omega(t) = (2 - \frac{1}{\pi}) + \frac{1}{\pi} \cos(\pi t + \pi t \sin(\pi t)) \text{ rad/s}$. Integrating again yields

$\theta - \frac{\pi}{6} = \frac{1}{\pi}[2 \sin(\pi t - \pi t - \pi t \cos(\pi t + 2\pi^3 t)]$

and

$\theta(t) = \frac{\pi}{6} + 2t + \frac{1}{\pi} (2 \sin(\pi t - \pi t - \pi t \cos(\pi t)) \text{ rad}$

1.95 Solution: $\omega_0 = 3 \text{ rad/s}$ and $\alpha(\omega) = -2\omega^2 \text{ rad/s}^2$. From eq. (1.87) $\alpha = \omega \frac{d\omega}{dt} = -2\omega^2$. Solving yields $\int_0^\omega \frac{d\omega}{\omega^2} = -2 \int_0^t d\theta$, or $\frac{\tan \theta}{2} = -2\theta$. $\omega = 3e^{-2\theta}$ so that $\frac{d\theta}{dt} = 3e^{-2\theta}$ or $\int_0^\theta e^{2\theta}d\theta = \int_0^t 3dt = 3t$. Evaluating the other integral yields

$\frac{1}{2} e^{2\theta} |_0^\theta = 3t$ or $e^{2\theta} - 1 = 6t$

and $2\theta = \ln(6t + 1)$ or $\theta(t) = \frac{1}{2} \ln(6t + 1)$ and $\theta(10) = \frac{1}{2} \ln(61) = 2.055 \text{ rad}$.

1.96 Solution: $\omega_0 = 2 \text{ rad/s}$ and $\alpha(\omega, t) = -0.01\omega + 4t \text{ rad/s}^2$. Then $\frac{d\omega}{dt} = -0.01\omega + 4t$ which is a first order differential equation of the form: $\dot{\omega} + 0.01\dot{\omega} = 4t$, $\omega_0 = 2$. Using the integrating factor $x(t) = e^{0.01t}$, equation (1.34) yields the solution

$\omega(t) = e^{-0.01t} [C + 40,000e^{0.01t}(0.0t - 1)]$

Since $\omega(0) = 2$, $C = -3998$ and

$\omega(t) = -3998e^{-0.01t} + 4000(0.01t - 1)$

$\omega(0) = -4000 - 3998e^{-0.01t} + 40t \text{ rad/s} = [0.01t - 1 + e^{-0.01t}](4 \times 10^4) \text{ rad/s}$

Since $\omega = \frac{d\theta}{dt}$, integrating this $\omega(t)$ yields $\theta(t)$.

$\theta(t) = 400 + [0.005t^2 - t - 0.01e^{-0.01t}](4 \times 10^4) \text{ rad}$

where 3998 has to be rounded to 4000.
1.97 The equation of motion can be written as \( \frac{d\omega}{dt} = 2 \sin \theta - 0.4\omega \). Since \( \omega = \frac{d\theta}{dt} \), this can be written as a second order nonlinear equation in \( \theta \):
\[
\ddot{\theta} + 0.4\dot{\theta} - 2\sin \theta = 0 \quad \theta(0) = \frac{\pi}{6} \text{ and } \dot{\theta}(0) = 0
\]
which can be solved by numerical integration as suggested in equation (1.93).
That is
\[
\begin{bmatrix}
\omega_{n+1} \\
\theta_{n+1}
\end{bmatrix} = \begin{bmatrix}
\omega_n + (2 \sin \theta_n - 0.4\omega_n) \Delta t \\
\theta_n + \omega_n \Delta t
\end{bmatrix}, \quad \begin{bmatrix}
\omega_0 \\
\theta_0
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{\pi}{6}
\end{bmatrix}
\]
which is plotted in the following Mathcad file:

\[
\begin{align*}
\text{i} &= 0 \ldots 500 \\
\Delta t &= 0.01 \\
\omega_0 &= 0 \\
\theta_0 &= \frac{\pi}{6}
\end{align*}
\]

\[
\begin{bmatrix}
\omega_{i+1} \\
\theta_{i+1}
\end{bmatrix} := \begin{bmatrix}
\omega_i + (2 \sin \theta_i - 0.4\omega_i) \cdot \Delta t \\
\theta_i - \omega_i \cdot \Delta t
\end{bmatrix}
\]

FIGURE S1.97

The Matlab code is:

```matlab
function xdot=onept97(t,x);
xdot=[x(2);2*sin(x(1))-0.4*x(2)];
command window:
EDU>tspan=[0 5];
EDU>x0=[pi/6;0];
EDU>ode45('onept97', tspan, x0);
```

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This is a second order, nonlinear differential equation in $\theta$ which can be solved numerically by using the first order form suggested in equation (1.93). From the given form of $\ddot{\theta} + 0.02|\dot{\theta}| - 3 \cos \theta = 0$, $\theta(0) = \pi/6$ and $\dot{\theta}(0) = 0$. Equation (1.93) becomes

$$\begin{bmatrix} \omega_{n+1} \\ \dot{\theta}_{n+1} \end{bmatrix} = \begin{bmatrix} \omega_n + (3 \cos \theta_n - 0.02\omega_n|\omega_n|\Delta t) \\ \dot{\theta}_n + \omega_n \Delta t \end{bmatrix}$$

which is plotted below in Mathcad:

In Matlab the code is:

```matlab
function xdot=onept98(t,x);
xdot=[x(2);3*cos(x(1))-0.02*x(2)*abs(x(2)));
command window
EDU>tspan=[0 5];
EDU>x0=[pi/6;0];
EDU>ode('onept98',tspan,x0);
```

![FIGURE S1.98](image-url)
1.99 Follow sample 1.18. Given \( \mathbf{r}(t) = 3\cos t\hat{i} + 3\sin t\hat{j} + 4t\hat{k} \) m, differentiation yields \( \mathbf{v}(t) = -3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k} \) and \( \mathbf{a}(t) = -3\cos t\hat{i} - 3\sin t\hat{j} \) so that \( \mathbf{v}(3) = \hat{i} + \hat{j} + 4\hat{k} \) and \( \mathbf{a}(3) = \hat{i} - \hat{j} \). The unit tangent vector \( \hat{e}_t \) is calculated from eq. (1.97) to be

\[
\hat{e}_t(3) = \frac{\mathbf{v}'(3)}{|\mathbf{v}'(3)|} = -0.085\hat{i} - 0.594\hat{j} + 0.8\hat{k}
\]

\( a_t(3) = (a \cdot \mathbf{e}_t)\hat{e}_t = 0 \), \( a_n = a - a_t = a \) so that

\[
\hat{e}_n = \frac{\hat{a}_n}{|\hat{a}_n|} = \frac{\hat{a}}{|\hat{a}|} = 0.99\hat{i} - 0.141\hat{j}
\]

\( \hat{e}_b = \hat{e}_t \times \hat{e}_n = 0.133\hat{i} + 0.792\hat{j} + 0.6\hat{k} \)

1.100 From problem 1.99

\( \mathbf{v}(t) = -3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k} \) and \( \mathbf{a}(t) = -3\cos t\hat{i} - 3\sin t\hat{j} \) m/s²

Note magnitude of both \( \mathbf{v}(t) \) and \( \mathbf{a}(t) \) is constant.

1.101 From equation 1.100, \( \rho(t) = \frac{v^2(t)}{|\mathbf{a}_n(t)|} \) where \( v^2(t) = \mathbf{v}(t) \cdot \mathbf{v}(t) = 9\sin^2 t + 9\cos^2 t + 16 = 9 + 16 = 25 \). Compute \( |\mathbf{a}_n(t)| \). From eq. (1.97) \( \hat{e}_t = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{5}(-3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k}) \). Then from eq. (1.98)

\[
\mathbf{a}_t(t) = (\mathbf{a} \cdot \mathbf{e}_t)\mathbf{e}_t = \frac{1}{5}[-3\cos t(-3\sin t) + (-3\sin t)(3\cos t) + (0)(4)]\mathbf{e}_t = 0.
\]

Now \( \mathbf{a}_n = \mathbf{a} - \mathbf{a}_t = \mathbf{a} - 0 = \mathbf{a} \). Thus \( |\mathbf{a}_n(t)| = |\mathbf{a}(t)| = \sqrt{3^2 \cos^2 t + 3^2 \sin^2 t} = 3 \). Thus

\[
\rho(t) = \frac{v^2(t)}{|\mathbf{a}_n(t)|} = \frac{1}{3}(25) = 8.33 \text{ m}, \text{ a constant so that the motion is circular, moving at a constant angular velocity.}
\]

1.102 Given \( \mathbf{r}(t) = t^2\hat{i} + 3t\hat{j} + 10\sin t\hat{k} \) m, successive differentiation yields the velocity and acceleration:

\[
\mathbf{v}(t) = \mathbf{\dot{r}}(t) = 2t\hat{i} + 3\hat{j} + 10\cos t\hat{k} \text{ m/s}
\]

\[
\mathbf{a}(t) = \mathbf{\ddot{r}}(t) = 2\hat{i} - 10\sin t\hat{k} \text{ m/s}^2
\]

From eq. (1.100) the radius of curvature \( \rho(t) = \frac{v^2(t)}{|\mathbf{a}_n|} \)

Now \( v^2 = v \cdot v = 4t^2 + 9 + 100\cos^2 t \). Following eq. (1.97) \( \dot{e}_t(t) = \frac{\mathbf{v}}{|\mathbf{v}|} \), \( \mathbf{a}_t = (\mathbf{a} \cdot \dot{e}_t)\dot{e}_t \), \( \mathbf{a}_n = \mathbf{a} - \mathbf{a}_t \) and \( \rho(t) = \frac{v^2}{|\mathbf{a}_n|} \). Programming these formulations and evaluating at each value of \( t \) yields

a) \( t = 1 \text{ s}, \rho(1) = 7.23 \text{ m} \)

b) \( t = 3 \text{ s}, \rho(3) = 126.645 \text{ m} \)

c) \( t = 5 \text{ s}, \rho(5) = 13.346 \text{ m} \)
The solution in Mathcad is given in the following:

\[
t = 5
\]

\[
v = \begin{bmatrix}
2 \cdot t \\
3 \\
-10 \cdot \cos(t)
\end{bmatrix}, \quad a = \begin{bmatrix}
2 \\
0 \\
-10 \cdot \sin(t)
\end{bmatrix}
\]

\[
et = \frac{v}{|v|} \quad at := (a \cdot et) \cdot et \quad an = a - at
\]

\[
\rho = \frac{v \cdot v}{|an|} \quad \rho = 13.346
\]

The Matlab code is

```matlab
t=5;v=[2*t;3;10*cos(t)];a=[2;0;-10*sin(t)];
et=v/norm(v);at=dot(a,et)*et;an=a-at;
pro=dot(v,v)/norm(an)
```

1.103 Solution: \( r(t) = \frac{1}{2}(1 + t^2) \) and \( \theta = \pi t^2 \) so that \( \dot{r}(t) = t \) and \( \ddot{r}(t) = 1, \dot{\theta} = 2\pi t \) and \( \ddot{\theta} = 2\pi \). From equation (1.103)

\[
v = \dot{r} \hat{e}_r + \dot{\theta} \hat{e}_\theta = t \hat{e}_r + \pi t (1 + t^2) \hat{e}_\theta
\]

\[
a = (\ddot{r} - \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{e}_\theta = \left(1 - 2\pi^2 t^2 (1 + t^2)\right) \hat{e}_r + \left(\pi (1 + t^2) + 4t^2 \pi \right) \hat{e}_\theta
\]

To plot the motion define \( t : 0, 0.1...2 \), define \( r = \frac{1+t^2}{2} \) and \( \theta = \pi t^2 \). Then let \( x = r \cos \theta \) and \( y = r \sin \theta \) which is plotted below.

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The Mathcad code is:

\[
\begin{align*}
  t &:= 0, 0.01..2 \\
  r(t) &= \frac{1 - t^2}{2} \\
  \theta(t) &= \pi t^2 \\
  x(t) &= r(t) \ 8 \cos(\theta(t)) \\
  y(t) &= r(t) \sin(\theta(t))
\end{align*}
\]

![Graph of the motion](image)

**FIGURE S1.103**

The Matlab code is:

```matlab
EDU>t=linspace(0,2);
EDU>r=(1+t^2)/2;th=pi*t.^2;
EDU>x=r.*8.*cos(th); y=r.*sin(th);
EDU>plot(x,y)
```

1.104 The value of \( v(t) \) is always positive. Note that \( |v(t)| = \sqrt{t^2 + \pi^2 t^2 (1 + t^2)^2} = t \sqrt{1 + \pi^2 (1 + t^2)^2} = ds/dt \) so that

\[
s - s_0 = \int_0^t \sqrt{t^2 + \pi^2 t^2 (1 + t^2)^2} dt = 18.977 \text{ m}.
\]

1.105 Solution: \( r(t) = 2 \) so that \( \dot{r} = \ddot{r} = 0 \) and there is circular motion. \( \theta(t) = \sin \pi t \) so that \( \dot{\theta} = \pi \cos \pi t \) and \( \ddot{\theta} = -\pi^2 \sin \pi t \). From equations (1.103):

\[
\begin{align*}
  v(t) &= r \dot{\theta} \hat{e}_\theta = 2\pi \cos \pi \theta \\
  a(t) &= -2\pi^2 \cos^2 \pi \dot{\theta} \hat{e}_r - 2\pi^2 \sin^2 \pi \theta \hat{e}_\theta
\end{align*}
\]

Let \( x = r \cos \theta \) and \( y = r \sin \theta \) to plot the motion as illustrated below.
The Mathcad solution is:

\[
\begin{align*}
    t & := 0, 0.01 \ldots 2 \\
    \theta(t) & := \sin \left( \pi \cdot t \right) \\
    r(t) & := 2
\end{align*}
\]

FIGURE S1.105

The Matlab code is:

\[
\begin{align*}
    \text{EDU} \backslash \text{> linspace}(0,2);
    \text{EDU} \backslash \text{> r=2;} \text{th} = \sin(\text{pi} \cdot t);
    \text{EDU} \backslash \text{> x=r*cos(th);} \text{y=5*sin(th)};
    \text{EDU} \backslash \text{> plot(x,y)}
\end{align*}
\]

1.106 Solution:

\[
\begin{align*}
    \frac{ds}{dt} = |v(t)| = \sqrt{4\pi^2 \cos^2 \pi t} = |2\pi \cos \pi t|
    s - s_0 = \int_0^t |2\pi \cos \pi t| dt = 8 \text{ m}
\end{align*}
\]

1.107 Since the particle starts from rest, \(\dot{r}(0) = 0\), \(\dot{\theta}(0) = 0\), \(r(0) = 1\), and \(\theta(0) = 0\). To determine \(v(t)\) and \(a(t)\) from eqs.(1.103) we need \(r(t)\), \(\theta(t)\), \(\dot{r}(t)\) and \(\dot{\theta}(t)\) which we can calculate by integrating \(\ddot{r}\) and \(\ddot{\theta}\)

\[
\begin{align*}
    \dot{r}(t) - \dot{r}(0) & = \int_0^t \ddot{r}(x) dx = \int_0^t 2e^{-x} dx = 2(1 - e^{-t}) \\
    r(t) - 1 & = 2 \int_0^t (1 - e^{-x}) dx \text{ so that } r(t) = 1 + 2(t + e^{-t} - 1) = -1 + 2t + 2e^{-t}
\end{align*}
\]

Likewise

\[
\begin{align*}
    \dot{\theta}(t) - 0 & = \pi \int_0^t dx = \pi t \text{ and } \theta(t) - 0 = \pi \int_0^t xdx = \frac{\pi t^2}{2}
\end{align*}
\]
Now from eqs. (1.103)

\[ v(t) = 2(1 - e^{-t})\hat{e}_r + [(2e^{-t} + 2t - 1)\pi t]\hat{e}_\theta \]

\[ a(t) = [2e^{-t} - (2e^{-t} + 2t - 1)\pi t^2]\hat{e}_r + [(2e^{-t} + 2t - 1)\pi + 2(2 - 2e^{-t})\pi t] \]

1.108 Let \( r(t) = 1 + 2(t + e^{-t} - 1) \) and \( \theta(t) = \frac{\pi t^2}{2} \). Then define \( x(t) = r(t)\cos\theta(t) \), \( y(t) = r(t)\sin\theta(t) \) and plot (Mathcad solution)

\[
\begin{align*}
    t &:= 0, 0.01 .. 2 \\
    r(t) &= 1 - 2\cdot\left[t - e^{-1} - 1\right] \\
    \theta(t) &= \frac{\pi t^2}{2} \\
    x(t) &= r(t)\cdot\cos\left(\theta(t)\right) \\
    y(t) &= r(t)\cdot\sin\left(\theta(t)\right)
\end{align*}
\]

**FIGURE S1.108**

The distance traveled is \( \int_0^2 |v(t)|\,dt = 14.438 \text{ m} \)

The Matlab code is:

EDU> linspace(0,2);
EDU> r=1+2*(t-exp(-t)-1);th=(pi*t.ˆ2)/2;
EDU> x=r.*cos(th);y=r.*sin(th);
EDU> plot(x,y)

1.109 Given \( \omega = 2\pi \text{ rad/s} \) and \( r(t) = r_0 + r_a\sin 2\pi t \), to determine the acceleration requires expressions for \( r, \theta, \dot{r} \) and \( \ddot{\theta} \). Since \( \omega = 2\pi, d\theta = 2\pi dt \) and \( \theta = \theta_0 + 2\pi t, \dot{\theta} = 2\pi \), and \( \ddot{\theta} = 0 \).

Likewise \( \dot{r} = 2\pi r_a\cos 2\pi t \) and \( \ddot{r} = -4\pi^2 r_a\sin 2\pi t \). From eq. 1.103, \( a_r = \ddot{r} - r\dot{\theta}^2 = -4\pi^2 r_a\sin 2\pi t - (r_0 + r_a\sin 2\pi t)4\pi^2 = -4\pi^2(2r_a\sin 2\pi t + r_0) \text{ m/s}^2 \)

\[ a_\theta = r\ddot{\theta} - 2r\dot{\theta}^2 = (2)(2\pi r_a\cos 2\pi t)(2\pi) = 8\pi^2 r_a\cos 2\pi t \text{ m/sec}^2 \]
1.110 Solution: a) Let $r_0 = 2$, $r_a = 1.5 < r_0$ and $\theta_0 = 0$, $r(t) = 2 + 1.5 \sin 2\pi t$, $\theta(0) = 0$ so $\theta(t) = 2\pi t$. For one revolution let $t = 0$, 0.01..1 sec, then let $x(t) = r(t) \cos \theta(t)$ and $y(t) = r(t) \sin \theta(t)$ which is plotted below using Mathcad

\[ t = 0, 0.01..2 \]

\[ r(t) := 2 + 1.5 \cdot \sin(2 \cdot \pi \cdot t) \]
\[ \theta(t) := 2 \cdot \pi \cdot t \]

\[ \begin{array}{c}
\text{FIGURE S1.110} \\
\end{array} \]

The Matlab code for this plot is

EDU>linspace(0,2);
EDU>r=2+1.5*sin(2*pi*t);th=2*pi*t;
EDU>x=r.*cos(th);y=r.*sin(th),
EDU>plot(x,y)

b) The velocity is $v(t) = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta$ so that:

\[ |v(t)| = \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2} = \pi \sqrt{1.5 \cos^2 2\pi t + (2 + 1.5 \sin 2\pi t)^2} \]

so $s = \int_0^1 |v(t)| dt = 14.407 \text{ m}$
1.111 Solution: \( a_r = 2t, \ a_\theta = \cos(\pi z) \), \( r(0) = .5 \ m, \ \theta(0) = 0 \). Since it starts from rest \( \dot{\theta}(0) = \dot{r}(0) = 0 \). From eq. (1.103) \( \ddot{r} - r\dot{\theta}^2 = 2t, \ r\ddot{\theta} + 2\dot{r}\dot{\theta} = \cos(\pi t) \) which is a system of 2 coupled 2nd order equations which are nonlinear and inhomogeneous. The initial conditions (4) are given above. To solve numerically follow sample 1.22 (use \( x = \dot{r}, \ y = \dot{\theta} \))

\[
\begin{align*}
\ddot{r} &= r\dot{\theta}^2 + 2t \\
\dot{\theta} &= -2\frac{\dot{r}}{r} + \frac{\cos \pi t}{r}
\end{align*}
\]

The Euler formula becomes (\( t_{n+1} = t_n + \Delta t \))

\[
\begin{align*}
\begin{bmatrix} x_{n+1} \\ r_{n+1} \\ y_{n+1} \\ \theta_{n+1} \end{bmatrix} &= \begin{bmatrix} (r_n \cdot y_n^2 + 2t_n)\Delta t + x_n \\ r_n + x_n\Delta t \\ (-2x_n \cdot y_n/r_n + \cos \pi t_n)\Delta t + y_n \\ y_n\Delta t + \theta_n \end{bmatrix} \\
\begin{bmatrix} x_0 \\ r_0 \\ y_0 \\ \theta_0 \end{bmatrix} &= \begin{bmatrix} 0 \\ .5 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

Once these are solved the polar coordinate \( r(t) \) and \( \theta(t) \) are given by the digital record for \( r_n \) and \( \theta_n \).
The trajectory is obtained by plotting $r_n \cos \theta_n$ vs. $r_n \sin \theta_n$ from above. The Mathcad solution is:

\[
\begin{align*}
  i &:= 0..2000 \quad \Delta t := 0.001 \\
  t &:= i \cdot \Delta t \\
  \alpha(v, r, \omega, \theta, t) &:= \frac{1}{r} \cdot \left(\cos\left(\pi \cdot t\right) - 2 \cdot v \cdot \omega\right) \\
  a(v, r, \omega, \theta, t) &:= 2 \cdot t + r \cdot \omega^2 \\

  \begin{bmatrix}
    v_0 \\
    r_0 \\
    \omega_0 \\
    \theta_0
  \end{bmatrix} &:=
  \begin{bmatrix}
    0 \\
    0.5 \\
    0 \\
    0
  \end{bmatrix} \\
  \begin{bmatrix}
    v_i + 1 \\
    r_i + 1 \\
    \omega_i + 1 \\
    \theta_i + 1
  \end{bmatrix} &:=
  \begin{bmatrix}
    a(v_i, r_i, \omega_i, \theta_i, t_i) \cdot \Delta t + v_i \\
    r_i + v_i \cdot \Delta t \\
    \omega_i + \alpha(v_i, r_i, \omega_i, \theta_i, t_i) \cdot \Delta t \\
    \theta_i + \omega_i \cdot \Delta t
  \end{bmatrix}

  x_i := r_i \cdot \cos\left(\theta_i\right) \quad y_i := r_i \cdot \sin\left(\theta_i\right)
\end{align*}
\]

FIGURE S1.112
The equivalent Matlab code is:

function xdot=onept112(t,x);
xdot=[x(2);x(2)*x(3)^2+2*x(4);-2*x(1)*x(3)/x(1)+(cos(pi*t))/x(2)];

In the command window:
EDU>tspan=[0 2];
EDU>x0=[0;5;0;0]
EDU>[t,x]=ode45('onept112',tspan,x0);
EDU>xc=x(:,2).*cos(x(:,4));ys=x(:,2).*sin(x(:,4));
EDU>plot(xc,ys)

1.113 The acceleration components in polar coordinates are given in eq. (1.04) to be

\[ a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \]

We are given \( \ddot{\theta} \) which we need to integrate to get \( \dot{r}, r, \theta \) and \( \dot{\theta} \). First consider \( \ddot{\theta} = 0 \) so that \( \dot{\theta} = \) constant = 1.5 rad/s. Integrating again yields \( \theta(t) = 1.5t \). Then the above becomes simply

\[ a_r = \ddot{r} - (1.5)^2r, \quad a_\theta = 2(1.5)\dot{r} = 3\dot{r} \]

Integrating \( \ddot{r} = 3 - 0.01\dot{r} \) requires the solution of

\[ \ddot{r} + 0.01\dot{r} = 3 \]

Which is a second order differential equation with particular solution \( r_P = \frac{3t}{0.01} \).

The homogeneous equation is \( \ddot{r} + 0.01\dot{r} = 0 \) which has solution \( r_1 = A \) and \( r_2 = Be^{\lambda t} \). Substitution yields

\[ \lambda^2 + 0.01\lambda = 0 \quad \text{or} \quad \lambda = -0.01 \] so the homogeneous solution is

\[ r_H(t) = A + Be^{-0.01t} \]

and the general solution is

\[ r = r_H + r_P = A + Be^{-0.01t} - 300t \]

To get \( A \) and \( B \) apply the initial condition \( r(0) = 0.4 \) and \( \dot{r}(0) = 0 \)

\[
\begin{align*}
    r(0) &= A + B = 0.4, \\
    \dot{r}(0) &= 0.01B + 300 = 0 \text{ or } B = -30,000 \\
    A &= 30,000.4
\end{align*}
\]

Thus

\[ r(t) = -30,000.4 - 30,000e^{-0.01t} + 300t \text{ m} \]

Thus \( \dot{r} = 300(1 - e^{-0.01t}) \), \( \ddot{r} = 3e^{-0.01t} \) so that

\[
\begin{align*}
    a_r &= 3e^{-0.01t} - 2.25(30000.4 - 30000e^{-0.01t} + 300t) \\
    a_\theta &= (3)(30000.4) - 90,000e^{-0.01t} + 900t
\end{align*}
\]
1.114 The trajectory is a part of \( x(t) = r \cos \theta \) versus \( y = r \sin \theta \). The form of \( r(t) \) and \( \theta(t) \) are given in the previous problem. Let \( t = 0, 0.1...4 \) and plot \( y \) vs. \( x(t) \) as given below from Mathcad

\[
\begin{align*}
t &= 0, 0.1..4 \\
r(t) &= 30000.4 - 30000 \cdot e^{-0.01 \cdot t} - 300 \cdot t \\
\theta(t) &= 1.5 \cdot t
\end{align*}
\]

![Figure S1.114](image)

The Matlab code is:

```matlab
EDU>t=linspace(0,4);
EDU>th=1.5*t;r=3000.4-30000*exp(-0.01*t)+300*t;
EDU>x=r.*cos(th);y=r.*sin(th);
EDU>plot(x,y)
```

1.115 In order to determine the velocity and acceleration from eq. (1.103), \( r(t), \dot{r}(t), \ddot{r}, \theta, \dot{\theta} \) and \( \ddot{\theta} \) are needed. Since \( \dot{\theta} = \pi/4 \) so that \( \ddot{\theta} = 0 \) and \( \theta(t) = (\pi/4)t \) (assuming \( \theta(0) = 0 \)). Then \( r(t) = r(\theta(t)) = 100 + 60 \cos \frac{\pi t}{4} \) so that \( \dot{r}(t) = -15\pi \sin \frac{\pi t}{4} \) and \( \ddot{r} = -\frac{15\pi^2}{4} \cos \frac{\pi t}{4} \).
Thus from eq. (103)
\[ \mathbf{v}(t) = (-15\pi \sin \frac{\pi t}{4})\hat{e}_r + \left(\frac{\pi}{4}\right)(100 + 60 \cos \frac{\pi t}{4})\hat{e}_\theta \]
and
\[ \mathbf{a}(t) = \left[-\frac{15}{4}\pi^2 \cos \frac{\pi t}{4} - \left(\frac{\pi}{4}\right)^2 \left(100 + 60 \cos \frac{\pi t}{4}\right)\right] \hat{e}_r + \left[-\frac{15}{2}\pi^2 \sin \frac{\pi t}{4}\right] \hat{e}_\theta \]
\[ \mathbf{a}(t) = -(100 + 120 \cos \frac{\pi t}{4})\] 
\[ \frac{\pi^2}{16} \hat{e}_r - \left[\frac{15\pi^2}{2}\right] \sin \frac{\pi t}{4} \hat{e}_\theta \]

1.116 Here we need to find \( r(t) \) from the drawing and knowledge of \( \theta(t) \). \( \theta(t) = \frac{\pi}{4} \sin \pi t \), so that \( \dot{\theta} = \frac{\pi^2}{4} \cos \pi t \) and \( \ddot{\theta} = -\frac{\pi^3}{4} \sin \pi t = -\pi^2 \theta(t) \). From the drawing
\[ r(t) = 300 \sec \theta(t) = 300 \sec \left(\frac{\pi}{4} \sin \pi t\right) \]
\[ \dot{r}(t) = 75\pi^2 \tan \left(\frac{\pi}{4} \sin \pi t\right) \sec \left(\frac{\pi}{4} \sin \pi t\right) \cos \pi t \]
\[ \ddot{r}(t) = 300 \left[\frac{\pi^2}{16} \sin^2 \pi t \tan \left(\frac{\pi}{4} \sin \pi t\right) + \frac{\pi^4}{16} \cos^2 \pi t \tan^2 \left(\frac{\pi}{4} \sin \pi t\right) - 1\right] \sec \left(\frac{\pi}{4} \sin \pi t\right) \]
Then
\[ \mathbf{v} = [75\pi^2 \tan \left(\frac{\pi}{4} \sin \pi t\right) \sec \left(\frac{\pi}{4} \sin \pi t\right) \cos \pi t] \hat{e}_r + [75\pi^2 \sec \left(\frac{\pi}{4} \sin \pi t\right) \cos \pi t] \hat{e}_\theta \]
The acceleration becomes (in terms of \( \theta \), \( \dot{\theta} \) and \( \ddot{\theta} \))
\[ \mathbf{a} = 300 \sec \theta [(2 \tan^2 \theta) \dot{\theta}^2 + \tan \theta \ddot{\theta}] \hat{e}_r + 300 \sec \theta [\ddot{\theta} + 2 \tan \theta \dot{\theta}^2] \hat{e}_\theta \]
Let $\theta$ be the angle between $r(t)$ and the 60 mm line. Let $\beta(t)$ be the angle between the 100 mm radius and the end of $r(t)$. From this triangle
\[ r(t) = 60 \cos \theta + 100 \cos \beta \]
and from the law of sines: (1) $60 \sin \theta = 100 \sin \beta$. Differentiation of the law of sines yields (2) $60 \cos \theta \dot{\theta} = 100 \cos \beta \dot{\beta}$. These last two expressions can be used to remove the $\beta$ dependence. Differentiating $r(t)$ and using (1) and (2) to remove the $\dot{\beta}$ term yields
\[ \dot{r} = -60 \sin \theta \dot{\theta} - 100 \sin \beta \dot{\beta} = -60 \cos \theta \tan \beta \dot{\theta} \]
where we note that eq. (1) can be used to remove the $\beta$ dependence in favor of $\theta$. Now $\dot{\theta} = \text{constant} = \pi$ is given, so $\dot{r}$ becomes
\[ \dot{r} = -60\pi \sin \theta - 60\pi \cos \theta \tan \beta, \text{ where } \beta = \sin^{-1}(0.6 \sin \theta). \]
From (1)
\[ \ddot{r} = -60\pi^2 \cos \theta + 60\pi^2 \sin \theta \tan \beta - \frac{60^2 \pi^2 \cos^2 \theta}{100 \cos^2 \beta} \]
which can be verified symbolically using one of the codes. With $r$, $\dot{r}$ and $\ddot{r}$ given, $\dot{\theta} = \pi$, so that $\theta = \pi t$ and $\ddot{\theta} = 0$, then $\mathbf{a}$ can be determined from equation (104):
\[ a_r = \ddot{r} - r \dot{\theta}^2 \text{ and } a_\theta = 2 \dot{r} \dot{\theta} = 2\pi \dot{r}. \]

The Mathcad code for plotting this follows:
\[
a_r(t) := \text{ddr}(t) - r(t) \cdot \pi^2 \\
a_\theta(t) := 2 \cdot \text{dr}(t) \cdot \pi
\]

FIGURE S1.117
1.118 From the problem statement $\alpha(t) = 0.5 \cos t$. Integrating yields $\omega(t) = 0.5 \sin t + \omega_0 = 0.5 \sin t$ since it starts from rest. Integrating again ($\theta_0 = 0$) yields $\dot{\theta}(t) = -0.5 \cos t + C = -0.5 \cos t + 0.5$. Now recall the formulation of the previous problem $\beta(t) = a \sin(0.6 \sin(\theta(t)))$. Then using the formulas developed in problem 1.117, $a_r$ and $a_\theta$ can be determined. These are illustrated in the figure where the derivatives are confirmed by symbolic computations.

\[ t = 0, 0.01, 0.02, \ldots, 7 \quad \theta(t) = -0.5 \cos(t) + 0.5 \quad \omega(t) = 0.5 \sin(t) \quad \alpha(t) = 0.5 \cos(t) \]

First symbolically calculate the time derivative of $r(t)$ and define it by $r'(t)$

\[
\begin{align*}
\beta(t) &= a \sin(0.6 \sin(\theta(t))) \\
r'(t) &= 60 \cos(\theta(t)) - 100 \cos(\beta(t))
\end{align*}
\]

Next symbolically calculate the second derivative and denote it by $r''(t)$

\[
\begin{align*}
r''(t) &= -60 \sin(\theta(t)) \cdot \frac{d}{dt} \theta(t) - 100 \sin(\beta(t)) \cdot \frac{d}{dt} \beta(t)
\end{align*}
\]

Now define the acceleration components in terms of $r$ and its derivatives, theta and omega:

\[
\begin{align*}
ar(t) &= r''(t) - r(t) \cdot \omega(t)^2 \quad a_\theta(t) = 2 \cdot r' \cdot \omega(t) - r(t) \cdot \alpha(t)
\end{align*}
\]

**FIGURE S1.118**
The acceleration and velocity in cylindrical coordinates are given in eq. (1.109) and (1.111) and require \( r, \dot{r}, \ddot{r}, z, \dot{z}, \ddot{z}, \dot{\theta}, \ddot{\theta} \). Here \( r = 1.5, \dot{\theta} = \pi \) and \( z = 0.5 \cos 2\pi t \) so that \( \dot{r} = \ddot{r} = 0, \dot{\theta} = 0, \ddot{\theta} = 0, \dot{z} = -\pi \sin 2\pi t \) and \( \ddot{z} = -2\pi^2 \cos 2\pi t \). Thus

\[
\mathbf{v} = 1.5\pi \hat{\theta} - \pi \sin(2\pi t) \hat{z} \text{ m/s}
\]

\[
\mathbf{a} = -1.5\pi^2 \hat{r} - 2\pi^2 \cos(2\pi t) \hat{z} \text{ m/s}^2
\]

The Mathcad code for generating this plot is:

\[
\begin{align*}
i &:= 0..40 \quad t := 0.1 \cdot i \\
x_i &:= 1.5 \cdot \cos(\pi \cdot t_i) \quad y_i := 1.5 \cdot \sin(\pi \cdot t_i) \quad z_i := 0.5 \cdot \cos(2 \cdot \pi \cdot t_i)
\end{align*}
\]

\text{FIGURE S1.119}

The Matlab code for forming this plot is given in the following file:

\[
i=(0:1:40);
\]

\[
x=1.5*cos(pi*i*0.1);y=1.5*sin(pi*i*0.1);z=0.5*cos(pi*i*0.1);
\]

\[
\text{plot3}(x,y,z)
\]
The distance traveled is calculated from \( \frac{ds}{dt} = |v| \) or \( s = s_0 + \int_0^t |v| \, dt \). Assuming the particle starts out \( s_0 = 0 \), \( s = \int_0^2 \sqrt{(1.5\pi)^2 + (\pi \sin 2\pi t)} \, dt = 10.398 \text{ m} \).

The total distance traveled can also be calculated using software. For example, the Mathcad code for computing this distance follows:

\[
\begin{align*}
t &:= 0, 0.01 \ldots 2 \\
v(t) &:= \begin{bmatrix} 0 \\ 1.5 \cdot \pi \\ -\pi \cdot \sin(2 \cdot \pi \cdot t) \end{bmatrix} \int_0^2 |v(t)| \, dt = 10.398
\end{align*}
\]

**FIGURE S1.120**

1.121 Solution: \( r(t) = R - \frac{\dot{z}}{\dot{h}} R_t = R - \frac{h - 0.1h t}{\dot{h}} R = R - R + 0.1Rt = 0.1Rt \) so that \( \dot{r} = \frac{R}{10} \) and \( \ddot{r} = 0 \). \( \theta(t) = 2\pi t \) so that \( \dot{\theta} = 2\pi \) and \( \ddot{\theta} = 0 \).

\( z(t) = h(1 - 0.1t) \) so that \( \dot{z} = -0.1h \) and \( \ddot{z} = 0 \).

Thus
\[
v = 0.1R\dot{e}_r + (0.1Rt)(2\pi)\dot{e}_\theta - 0.1h\dot{e}_z = 0.1R\dot{e}_r + 0.2\pi R t \dot{e}_\theta - 0.1h \dot{e}_z, \text{ and}
\]
\[
a = (-0.1R t)4\pi^2\dot{e}_r + (\frac{R}{90} \cdot 2\pi)\dot{e}_\theta = -0.4R\pi^2\ddot{e}_r + 0.4\pi R \dot{e}_\theta.
\]

The particle reaches the bottom of the cone when \( z(t) = 0 \) or \( h(1 - 0.1t) = 0 \) or at \( t = 10 \text{ sec} \). Since \( \theta = 2\pi t \), as 10 sec have past, then \( \theta \) has gone around 10 times before it reaches the bottom (\( 2\pi10 = 20\pi \) or 10 complete cycles).

1.122 Solution: for \( R = 3 \text{ m} \) and \( h = 5 \text{ m} \), \( v(t) = 0.3\dot{e}_r + 0.6t \dot{e}_\theta - 0.5\dot{e}_z \) so that \( |v| = \sqrt{(0.3)^2 + (0.6t)^2 + (0.5)^2} \) as it takes 10 sec to travel to the bottom

\[
s = \int_0^{10} |v| \, dt = \int_0^{10} \sqrt{(0.3)^2 + (0.6t)^2 + (1.5)^2} \, dt = 94.67 \text{ m}
\]

1.123 From eq. (1.111) the given form of the differential equations are \( \dot{e}_r : 3 = \ddot{r} - r \dot{\theta}^2 (1) \), \( \dot{e}_\theta : 2r \ddot{\theta} = 0.1 \) (2) and from \( \dot{e}_z : \ddot{z} = -2 \). Since the particle starts from rest at zero, all the initial conditions are zero and since the coordinate \( z \) is decoupled it can be directly integrated to yield \( \dot{z} = -2t \) and \( \ddot{z} = -t^2 \). Equations (1) and (2) for \( r \) and \( \theta \) on the other hand are two coupled nonlinear equations which must be solved numerically. They are

\[
\begin{align*}
\ddot{r} &= r \dot{\theta}^2 + 3 \\
\ddot{\theta} &= -2 \frac{\ddot{r}}{r} + \frac{0.1}{r}
\end{align*}
\]
Putting these in 1st order form or Euler integration yields

\[
\begin{bmatrix}
\dot{r} \\
r \\
\dot{\theta} \\
\theta
\end{bmatrix} =
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
\dot{r} \\
r \\
\dot{\theta} \\
\theta
\end{bmatrix} =
\begin{bmatrix}
0.1/r - 2drd\theta/r \\
r\dot{\theta}y^2 + 3/dr \\
0.1/r - 2drd\theta/r \\
0.1/r - 2drd\theta/r
\end{bmatrix}
\]

The Euler equations then becomes (follow sample 1.22) as illustrated in

\[
\begin{bmatrix}
{r_{n+1}} \\
{dr_{n+1}} \\
{\theta_{n+1}} \\
{d\theta_{n+1}}
\end{bmatrix} =
\begin{bmatrix}
{r_n + dr_n \cdot \Delta t} \\
{(r_n \cdot d\theta_n^2 + 3) \cdot \Delta t + dr_n} \\
{\theta_n + d\theta_n \cdot \Delta t} \\
{d\theta_n + \left(\frac{0.1}{r_n} - 2\frac{dr_n \cdot d\theta_n}{r_n}\right) \cdot \Delta t}
\end{bmatrix},
\begin{bmatrix}
{r_0} \\
{dr_0} \\
{\theta_0} \\
{d\theta_0}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Figure S1.123 shows the integration along with a plot of the first 3 s using Mathcad.

\[
i = 0..3000 \\
\Delta t = 0.001 \\
t_i = \Delta t \cdot i \\
ar(r, dr, \theta, d\theta) := 3 + r \cdot d\theta^2 \\
a\theta(r, dr, \theta, d\theta) = \frac{1}{r} \cdot (0.1 - 2 \cdot dr \cdot d\theta) \\
r_0 = 1 \quad dr_0 = 0 \quad \theta_0 = 0 \quad d\theta_0 = 0
\]

\[
\begin{bmatrix}
{dr_i - 1} \\
{r_i - 1} \\
{d\theta_i - 1} \\
{\theta_i + 1}
\end{bmatrix} =
\begin{bmatrix}
{dr_i - ar(r_i, dr_i, \theta_i, d\theta_i) \cdot \Delta t} \\
{r_i - dr_i \cdot \Delta t} \\
{d\theta_i - a\theta(r_i, dr_i, \theta_i, d\theta_i) \cdot \Delta t} \\
{\theta_i - d\theta_i \cdot \Delta t}
\end{bmatrix}
\]
The required Matlab code is as follows:

```matlab
function xdot=onept123(t,x);
xdot=[x(2);3+x(1)*x(4)^2;x(4);(0.1-2*x(2)*x(4))/x(1)],
```

In the command window:

EDU>tspan[0 3];
EDU>0=[1;0;0;0];
EDU>[t,x]=ode('onept123',tspan,x0);
xc=x(:,1).*cos(x(:,3));yc=x(:,1).*sin(x(:,3));
plot(xc,yc)
```

1.124 Since \( R(t) = 0.3 + 0.1t^2 \), \( \dot{R}(t) = 0.2t \) and \( \ddot{R}(t) = 0.2 \). Since \( \theta(t) = \pi \sin(\pi t) \), \( \dot{\theta}(t) = \pi^2 \cos \pi t \) and \( \ddot{\theta}(t) = -\pi^3 \sin \pi t \). Since \( \phi(t) = \frac{\pi}{2} te^{-t} \), \( \dot{\phi}(t) = \frac{\pi}{2} (e^{-t} - te^{-t}) \) and \( \ddot{\phi}(t) = \frac{\pi}{2} (-e^{-t} + (-e^{-t} + te^{-t})) = \frac{\pi}{2} e^{-t}(t - 2) \). Now from equation (1.116)

\[
\mathbf{v}(t) = 0.2t \dot{\mathbf{e}}_R + \frac{\pi}{2} (0.3 + 0.1t^2)e^{-t}(1 - t) \dot{\mathbf{e}}_\phi + [0.3 + 0.1t^2][\sin(\frac{\pi t}{2} e^{-t})](\pi^2 \cos \pi t) \dot{\mathbf{e}}_\theta
\]

From Eq. (1.118)

\[
\mathbf{a}(t) = [0.2 - (0.3 + 0.1t^2)(\frac{\pi}{2} (1 - t)e^{-t})^2 - (0.3 + 0.1t^2)[\sin(\frac{\pi t}{2} e^{-t})]^2(\pi^4 \cos^2 \pi t)] \dot{\mathbf{e}}_R
\]

\[
+ [(0.3 + 0.1t^2)(\frac{\pi}{2} (t - 2)e^{-t}) + 2(0.2t)(\frac{\pi}{2} e^{-t}(1 - t)) - (0.3 + 0.1t^2)(\pi^2 \cos \pi t)^2 \sin(\frac{\pi t}{2} e^{-t}) \cos(\frac{\pi t}{2} e^{-t})] \dot{\mathbf{e}}_\phi
\]

\[
+ [(0.3 + 0.1t^2)(-\pi^3 \sin \pi t) \sin(\frac{\pi t}{2} e^{-t}) + 0.4t(\pi^2 \cos \pi t) \sin(\frac{\pi t}{2} e^{-t})] + 2(0.3 + 0.1t^2)(\frac{\pi}{2} e^{-t}(1 - t))(\pi^2 \cos \pi t) \cos(\frac{\pi t}{2} e^{-t}) \dot{\mathbf{e}}_\theta
\]

1.125 From Sample Problem 1.24 we have \( \mathbf{a} = g \sin \phi \dot{\mathbf{e}}_\phi \), and we are given that \( R = 200 \ mm, \ (R\theta)_0 = 600 \ mm/s \) and that \( \phi_0 = \pi/2, \ \theta_0 = 0, \ g = 9810 \ mm/s^2 \). Since \( h \) is constant \( (R = 200 \Rightarrow \dot{\theta}_0 = 3) \), \( h = \dot{\theta}_0 \sin^2 \phi_0 = 3 \ rad/s \). Now from the geometry of spherical coordinates the relation to rectangular coordinates is (from fig. 1.21)

\[
x(t) = R \sin \phi \cos \theta, \ y(t) = R \sin \phi \sin \theta, \ z(t) = R \cos \phi
\]

Now to solve the problem we need to integrate (via Eulerian) the two coupled equations

\[
\dot{\theta} = \frac{h}{\sin^2 \phi} \quad \text{and} \quad \ddot{\theta} = \frac{h^2 \cos \phi}{\sin^2 \phi} + \frac{g}{R} \sin \phi
\]

subject to the initial condition \( \phi_0 = \frac{\pi}{2}, \ \theta_0 = 0, \ \dot{\phi}_0 = 0 \).
The Euler formulation and a plot of the motion is given in the following Mathcad code:

\[ i = 0..800 \quad \Delta t = 0.001 \quad t_i = \Delta t \cdot i \quad h = 3 \quad R = 200 \quad g = 9810 \]

\[ a\phi(\phi) = \frac{h^2 \cdot \cos(\phi)}{(\sin(\phi))^2} - \frac{g}{R} \cdot \sin(\phi) \]

\[ d\theta(\phi) = \frac{3}{(\sin(\phi))^2} \]

\[ d\phi_0 = 0 \quad \phi_0 = \frac{\pi}{2} \quad \theta_0 = 0 \]

\[ \begin{bmatrix} d\phi_i - 1 \\ \phi_i - 1 \\ \theta_i - 1 \end{bmatrix} = \begin{bmatrix} d\phi_i - a\phi(\phi_i) \cdot \Delta t \\ \phi_i - d\phi_i \cdot \Delta t \\ \theta_i + d\theta(\phi_i) \cdot \Delta t \end{bmatrix} \]

\[ x_i = R \cdot \sin(\phi_i) \cdot \cos(\theta_i) \quad y_i = R \cdot \sin(\phi_i) \cdot \sin(\theta_i) \quad z_i = R \cdot \cos(\phi_i) \]

**FIGURE S1.125**
The Matlab code is (using an Euler method):

```matlab
x(1)=0;p(1)=pi/2;th(3)=0;h=3;R=200;g=9810
dt=0.001; for n=1:800;
x(n+1)=x(n)+(h^2*cos(p(n))/(sin(p(n))^2 + (g/R)*sin(p(n)))*dt;
p(n+1)=p(n)+x(n)*dt;
end
xc=R*sin(p).*cos(p);yc=R*sin(p).*sin(p);zc=R*cos(p);
plot3(xc,yc,zc)
```

1.126 This is a repeat of problem 1.125 for the case that \( \dot{\theta}_0 = 0 \). Since \( \dot{\theta}_0 \) and the constant of motion \( \dot{\theta} \sin^2 \phi = h \) must hold for all \( \theta \), \( h \) must be zero and the kinematic equations become \( \ddot{\phi} = \frac{gR}{\sin \phi} \) and \( \dot{\theta} = 0 \) which are numerically integrated in the following figure (Mathcad):

\[
\begin{aligned}
\dot{x}_i &= R \cdot \sin(\theta_i) \cdot \cos(\phi_i) \\
\dot{y}_i &= R \cdot \sin(\theta_i) \cdot \sin(\phi_i) \\
\dot{z}_i &= R \cdot \cos(\theta_i)
\end{aligned}
\]
The Matlab code is contained in the following file which when run will produce an identical plot:

```matlab
x(1)=0;p(1)=pi/2;th(1)=0;h=3;R=200;g=9810;
dt=0.001
for n=1:800;
    x(n+1)=x(n)+(g/R)*sin(p(n))*dt;
    p(n+1)=p(n)+x(n)*dt;
    th(n+1)=th(n);
end
xc=R*sin(p).*cos(p);yc=R*sin(p).*sin(p);ac=R*cos(p);
plot3(xc,yc,zc)
```

1.127 From the solution to problem 1.126, \( \ddot{\phi} = \frac{g}{R} \sin \phi \) for the case that the initial velocity in the circumferential direction is zero (\( h = 0 \)). We are given that \( \phi = \pi + \beta \), \( \beta \) small so that using the trig identity for \( \sin(\pi + \beta) \) we have \( \sin \phi = \sin(\pi + \beta) = \sin \pi \cos \beta + \cos \pi \sin \beta = -\sin \beta \). But since \( \beta \) is assumed small, we can use the small angle approximation: \( -\sin \beta = -\beta \), and our differential equation becomes \( (\ddot{\phi} = \frac{d^2}{dt^2}(\pi + \beta) = \ddot{\beta}) \)

\[ \ddot{\beta} + \frac{g}{R} \beta = 0 \]
which was solved analytically in Sample Problem 1.9. The solution is

\[ \beta(t) = \beta_0 \cos(\sqrt{\frac{g}{R}}t) + \sqrt{\frac{g}{R}} \dot{\beta}_0 \sin(\sqrt{\frac{g}{R}}t) \]

where \( \beta_0 \) and \( \dot{\beta}_0 \) are the initial angle and angular velocity given to the particle.

1.128 From the value of \( a \) in spherical coordinates, the following 3 equations result

\[
12 = \ddot{R} - R\ddot{\phi}^2 - R\dot{\theta}^2 \sin^2 \phi \\
4 = R\ddot{\phi} + 2R\dot{\phi} \ddot{\phi} - R\dot{\theta}^2 \sin \phi \cos \phi \\
5 = R\ddot{\theta} \sin \phi + 2R\dot{\theta} \dot{\phi} \sin \phi + 2R\dot{\phi} \dot{\theta} \cos \phi
\]

which can be numerically integrated subject to the initial conditions

\[
R(0) = 2 \quad \dot{R}(0) = 2 \\
\theta(0) = 0 \quad \dot{\theta}(0) = -2 \quad (\text{since } v_\theta = -4 = R(0) \sin \phi(0) \dot{\theta}(0)) \\
\phi(0) = \pi/2 \quad \dot{\phi}(0) = 1/2 \quad (\text{since } v_\phi = R(0) \dot{\theta}(0) + 2 \dot{\theta}(0) + 1)
\]

by direct comparison with \( v(0) = 2\dot{e}_r + \dot{e}_\phi - 4\dot{e}_\theta \).
Once one numerically integrated, the expressions $x = R \sin \phi \cos \theta$, $y = R \sin \phi \sin \theta$, and $z = R \cos \phi$ can be used to construct a 3-D plot of the motion as described in the figure that follows using Mathcad.

\[
\begin{align*}
\Delta t &= 0.001, \\
\alpha_i &= 0..1000 \\
\frac{v_{R_i}}{R_i} &= 2 \\
\frac{v_{\phi_i}}{\phi_i} &= 2 \\
\frac{v_{\theta_i}}{\theta_i} &= 2 \\
\frac{v_{R_i}}{R_i} &= -2 \\
\frac{v_{\phi_i}}{\phi_i} &= -2 \\
\frac{v_{\theta_i}}{\theta_i} &= 0
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{array}{c}
\alpha_i \\
\frac{v_{R_i}}{R_i} \\
\frac{v_{\phi_i}}{\phi_i} \\
\frac{v_{\theta_i}}{\theta_i}
\end{array} \right] &= \left[ \begin{array}{c}
\frac{v_{R_i}}{R_i} = \alpha_i \\
\frac{v_{R_i}}{R_i} + \frac{v_{R_i}}{R_i} \cdot \Delta t \\
\frac{v_{\phi_i}}{\phi_i} - \alpha_i \\
\frac{v_{\phi_i}}{\phi_i} - \frac{v_{\phi_i}}{\phi_i} \cdot \Delta t \\
\frac{v_{\theta_i}}{\theta_i} - \alpha_i \\
\frac{v_{\theta_i}}{\theta_i} - \frac{v_{\theta_i}}{\theta_i} \cdot \Delta t
\end{array} \right]
\end{align*}
\]

\[
\begin{align*}
x_i &= R_i \sin \phi_i \cos \theta_i, \\
y_i &= R_i \sin \phi_i \sin \theta_i, \\
z_i &= R_i \cos \phi_i
\end{align*}
\]

**FIGURE S1.128**
The required Matlab code is:
R(1)=2;vR(1)=2;p(1)=pi/2; vp(1)=0.5; th(1)=0;vth(1)=-2; 

dt=0.001
for n=1:1000;
R(n+1)=R(n)+vR(n)*dt
vp(n+1)=vp(n)+((1/R(n))*4-2*vR(n)*vp(n)+R*vth(n)^2*sin(p(n)))*dxdt;
p(n+1)=p(n)+vp(n)*dt;
vth(n+1)=vth(n+1)+(1/(R(n)*sin(p(n)))*(5-2*vR(n)*vp(n)*sin(p(n)) 

-2*R(n)*vp(n)*vth(n)*cos(p(n)))*dxdt;

th(n+1)=th(n)+vth(n)*dt;
vR(n+1)=vR(n)+(12+R(n)*vp(n)^2+R(n)*vth(n)^2)*sin*p(n))+dt;
end
x=R.*cos(p).*sin(th);y=R.*sin(p).*sin(th);z=R.*cos(th);
plot3(x,t,z)

1.129 We are given \( x_A = 2t + 6 \) and \( x_{B/A} = 3t^2 + 2t + 3 \). From equation (1.120) \( x_B = x_{B/A} + x_A = 3t^2 + 4t + 9 \) so that \( \dot{x}_B = v_B = (6t + 4) \, \text{m/s} \), and \( a_B = 6 \, \text{m/s}^2 \). Likewise \( v_{B/A} = 6t + 2 \) and \( a_{B/A} = 6 \, \text{m/s}^2 \) from straightforward differentiation.

1.130 Since the particles start from rest \( v_A(0) = v_B(0) = 0 \) and since \( v_{B/A}(0) = v_B - v_A, v_{B/A}(0) = 0 \). Since \( x_A = 3t^2 + 2 \) and \( x_B = 6t^2 + 2 \), \( v_A = 3 \) and \( a_A = 0 \), \( v_B = 12t \) and \( a_B = 12 \). Since \( a_B \) and \( a_A \) are both constant and different in value: the two particles never have the same acceleration. The two particles reach the same position when \( x_{B/A} = 0 \). \( x_{B/A} = 6t^2 + 2 - (3t + 2) = 6t^2 - 3t = 0 \) when \( t = 0 \) and when \( t = 1/2s \). Since \( x_{B/A} = 6t^2 - 3t \), \( v_{B/A} = 12t - 3 \) so that they have the same velocity when \( t = 3/12 \) or \( t = 1/4s \).

1.131 Here you must pick \( t_0 \) carefully as the cars do not start moving at the same time. Car \( B \) moves with a constant velocity of \( v_B = 15 \, \text{m/s} \) so that \( x_B = 15t + x_B(0) \) for all time. Car \( A \) on the other hand, starts moving when car \( B \) crosses a position 100 meters out. Starting the clock at 0 when car \( B \) crosses the 100 m mark and taking the position \( x = 0 \) m as the reference point we can write \( x_B = 15t + 100, \) since \( v_B = 15 \, \text{m/s} \) implies that \( x_B(t) = 15t + x_B(0) = 15t + 100. \) Now consider car \( A \). We have that \( x_A(t) = t^2 \) until it reaches the speed of 20 m/s. This happens when \( 2t = 20 \) or at \( t = 10 \) sec. Note that \( x_A(10) = 100 \, \text{m}, \)
so that the cars cannot meet until after \( t > 10 \) s (i.e., car \( B \) started 100 m out). For \( t > 10 \) s, we are given that \( v_A = 20 \) m/s. Integrating yields

\[
\int_{x_A(10)}^{x_A} dx = \int_0^t 20 dt \quad \text{or} \quad x_A(t) = 20t - 200 + x_A(10)
\]

But \( x_A(10) = 100 \). Then for \( t > 10 \), \( x_A(t) = 20t - 100 \). Now for \( t > 10 \), \( x_{B/A} = 15t + 100 - (20t - 100) = -5t + 200 \). Then \( x_{B/A} = 0 \) yields \( t_m = 40 \) s, the time at which the cars meet. They have traveled at distance of \( x_A(40) = (20)(40) - 100 = 700 \) m which can be checked by calculating \( x_B(40) = (15)(40) + 100 = 700 \) m.

1.132 Particle \( B \) reverses direction when the velocity is zero. Since \( x_B = 12 + 18t - 4.9t^2 \), \( v_B = 18 - 9.8t = 0 \) at \( t_r = \frac{18}{9.8} = 1.83 \) s, at this time particle \( B \) changes directions. The two particles will meet when \( x_A = x_B \) or \( 5+2t = 12+18t-4.9t^2 \), which has solution \( t_c = -0.391 \), and 3.656s. So they collide at \( t_c = 3.656 \) s. With relative velocity \( v_{B/A}(t_c) = 16 - 9.8t = -19.83 \) m/s. Since \( t_c > t_r \), the particles collide after \( B \) reverses direction as illustrated in the figure.

\[
\begin{align*}
\text{t} & : 0, 0.1 \ldots 4 \\
x_B(t) & = 12 - 18 \cdot t - 4.9 \cdot t^2 \\
x_A(t) & = 5 - 2 \cdot t
\end{align*}
\]

**FIGURE S1.132**

1.133 Let car \( A \) be moving to the right with \( v_A(0) = 60 \) mph = 88 ft/s, \( x_A(0) = 0 \), and \( a_A = -16 \) ft/s\(^2\). Let car \( B \) be moving to the left with \( x_B(0) = 300 \), \( v_B(0) = -88 \) ft/s and \( a_B = 15 \) ft/s\(^2\). Integrating each yields

\[
\begin{align*}
v_A &= v_A(0) - \int_0^t 16 dt = 88 - 16t \\
x_A &= x_A(0) + 88t - 8t^2 = 88t - 8t^2 \\
v_B &= v_B(0) + 16 \int_0^t dt = -88 + 16t \\
x_B &= x_B(0) - 88t + 8t^2 = 300 - 88t + 8t^2
\end{align*}
\]
If they collide \( x_B = x_A \) or \( 300 - 88t + 8t^2 = -8t^2 + 88t \) or \( t = 2.109 \) and 8.891.

Now check to see if the cars are still moving at that time. The cars come to rest at \( v_A = v_B = 0 \) or \( 88 - 16t = 0 \) or \( t = \frac{88}{16} = 5.5 \) sec. Thus the cars hit at \( t = 2 \) sec. At that time

\[
v_{B/A}(t) = v_B(2.109) - v_A(2.109) = (-54.256) - (54.256) = 108.5 \text{ ft/s}.
\]

1.134 Solution: \( v_A(0) = 45 \text{ mph} = 66 \text{ ft/s}, v_B(0) = 58 \text{ mph} = 85 \text{ ft/s}, x_{B/A}(0) = 500 \text{ ft}, \) taking \( x_A(0) = 0 \) then \( x_B(0) = 500 \text{ ft}, a_A = 4 \text{ ft/s} \) and \( a_B = -2 \text{ ft/s} \). Integrating each yields

\[
v_A(t) = v_A(0) + 2t = 66 + 2t, \quad x_A(t) = x_A(0) + 66t + t^2 = 66t + t^2,
\]

\[
v_B(t) = v_B(0) - 2t = 85 - 2t, \quad x_B(t) = x_B(0) + 85t - t^2 = 500 + 85t - t^2.
\]

The cars meet when \( x_{B/A} = 0 \) or when \( 500 + 85t - t^2 = 0 \). This yields two roots for \( t \), one is negative and the other is \( t = 33.795 \text{s} \). Using the above formulas, the cars meet at

\[
x_A(33.795) = 3.373 \text{ ft} \sim 0.64 \text{ miles} \text{ and their velocities are}
\]

\[
v_A(33.795) = 133.59 \text{ ft/s} \text{ and}
\]

\[
v_B(35.795) = 17.41 \text{ ft/s}.
\]

1.135 Given: \( a_{B/A} = 3 \cdot \text{m/s}^2, v_{B/A}(0) = 0, x_{B/A}(0) = 10 \text{ m}, \) and \( x_A = 3t^2 + 1 \) compute \( a_B \). Differentiation of \( x_A \) yields \( v_A = 6t \) and \( a_A = 6 \). So that \( a_{B/A} = 3 = a_B - a_A = a_B - 6 \), and solving yields \( a_B = 9 \text{ m/s}^2 \). Integrating \( a_{B/A} = 3 \) yields \( v_{B/A}(t) = v_{B/A}(0) + 3t = 3t \). Integrating again yields \( x_{B/A} = v_{B/A}(0) + 3t^2 = 10 + 3t^2 \). Since \( x_{B/A} = x_B - x_A \) we have \( 10 + \frac{3t^2}{2} = x_B - 3t^2 - 1 \) or \( x_B(t) = 4.5t^2 + 11 \text{ m} \).

1.136 Since the time is so large in seconds, leave this in mph, miles and hours. For the 747, \( v_B = 575 \text{ mph} \) and \( x_B = 575t \text{ miles} \). At \( t = 3 \text{ hr.}, x_B(3) = 575(3) \). At \( t = 3 \) the Concorde takes off and \( v_A = 1336 \text{ mph} \) so that \( x_A = 1336t \text{ miles} \). For \( t \geq 3 \), \( x_B \) can be rewritten as \( x_B = (575)(3) + 575t, t \geq 3 \). The Concorde catches the 747 at \( x_A = x_B \) or \( (575)(3)+575t = 1336t \) or \( t = \frac{575(3)}{(70)} = 2.27 \text{ hrs} \).

1.137 The pendulum falls as \( \theta_B(t) = \theta_0 \cos \frac{\sqrt{g}}{t}, \) with \( \dot{\theta}_B(0) = 0 \). Consider the vertical \( (\theta = 0) \) position as the reference point at which we would like to know the time \( t_c \), i.e., solving \( \theta_0 \cos \sqrt{\frac{g}{t}} = 0 \) for \( t \) yields \( \sqrt{\frac{g}{t}} = \frac{\pi}{2} \) or \( t_c = \frac{\pi^2}{2g} = 1.121 \text{ s} \), the time it takes the pendulum to cover the angular distance from 20° to zero (note this does not depend on the value of \( \theta_0 \)). Next consider particle A. It
starts at rest \((x_A(0) = 0)\) 10 meters to the left, so that \(x_A(0) = -10 \text{ m}\). The particle’s acceleration can be written as \(a_n = a \text{ m/s}^2\) where \(a\) is an unknown constant. Integrating yields \(v_A = v_A(0) + at = at\). Integrating again yields: \(x_A(t) = x_a(0) + \frac{a}{2}t^2 = -\frac{a}{2}t^2 - 10\). The value at collision for \(t\) will be \(t_c = \frac{\pi}{2} \sqrt{\frac{g}{\ell}}\) so that \(\frac{a}{2}t^2 - 10 = 0\) or \(a = \frac{20}{t_c^2} = \frac{80g}{\pi^2\ell} = 15.9 \text{ m/s}^2\).

1.138 From eq. (1.77) \(v_B = ℓ\dot{θ}_B = 5\dot{θ}_B\) since the pendulum is in uniform circular motion or radius \(ℓ = 5 \text{ m}\). Differentiation of the expression for \(θ_B(t)\) yields \(\dot{θ}_B = -θ_0 \sqrt{\frac{g}{ℓ}} \sin\left(\sqrt{\frac{g}{ℓ}} t\right)\) so that \(v_B = ℓ\dot{θ} - θ_0 \sqrt{gℓ} \sin\left(\sqrt{\frac{g}{ℓ}} t\right)\). Let \(t_c\) be the time of impact calculated in problem 1.37. Then from 1.37, \(v_A = at_c\) so that \(v_{B/A} = -θ_0 \sqrt{gℓ} \sin\left(\sqrt{\frac{g}{ℓ}} t\right) - \frac{20}{t_c} t_c\). Substitute \(t_c = \frac{π}{2} \sqrt{\frac{g}{ℓ}}\) and \(θ_0 = \frac{π}{2}\) radians \(\left(20°\right)\) yields \(v_{B/A} = \frac{π}{2} \sqrt{gℓ} + \frac{40}{π} \sqrt{\frac{g}{ℓ}} = 20.279 \text{ m/s}\).

1.139 The acceleration of each particle is given by

\[
a_A = -9.81 - 0.1v_A
\]

(1)

\[
a_B = -9.81 - 0.1v_B
\]

(2)

with initial conditions \(v_A(0) = 30 \text{ m/s}, x_A(0) = 100 \text{ m}\) \(v_B(0) = x_B(0) = 0\). Equation (1) becomes \(\ddot{x}_A + 0.1\dot{x}_A = -9.81\) and (2) becomes \(\ddot{x}_B + 0.1\dot{x}_B = -9.81\)

These are second order, uncoupled linear differential equations with analytical solutions. The homogeneous solutions are \(x_A = A\), a constant and \(x_A = Be^{0.1t}\) or \(x_A(t) = A + Be^{-0.1t}\). The particular solution is \(Ct\). Substitution of \((x_A)_p = Ct\) into (1) yields \(C = -98.1\). Thus \((x_A)_p = -98.1t\) and the total solution is \(x_A = A + Be^{-0.1t} - 98.1t\). Applying the initial condition yields

\[100 = A + B\]

from the initial position

\[30 = -0.1B - 98.1\]

from the initial velocity

Solving yields \(B = -681\) and \(A = 781\). Thus

\[x_A(t) = 781 - 681e^{-0.1t} - 98.1t\frac{\text{m}}{\text{s}}\text{ and }v_A(t) = 68.1e^{-0.1t} - 98.1\]

Next consider particle \(B\) which has the same form but different initial conditions, i.e., \(x_B = A' + B'e^{-0.1t} - 98.1t\). The initial conditions yield

\[0 = A' + B'\]

\[0 = -0.1B' - 98.1\]

Solving yields \(A' = 981\) and \(B' = -981\) so that

\[x_B(t) = 981(1 - e^{-0.1t}) - 98.1t\frac{\text{m}}{\text{s}}\text{ and }v_B(t) = 98.1e^{-0.1t} - 98.1\]

Computing \(x_{B/A}(t)\) and setting this equal to zero yields

\[0 = 981 - 781 + (-981 + 681)e^{-0.1t_c}\]

Solving yields \(t_c = \frac{\ln(2/3)}{-0.1} = 4.0555\), the time of collision. The relative velocity at \(t_c = 4.055\) is \(v_B(4.055) - v_A(4.055) = 20 \text{ m/s}\)
The two differential equations for the acceleration of the particles are separable and can be solved by hand. However, to gain a better conceptual understanding of the particle motion, the equations are solved by numerical integration using the following Mathcad code:

\[
\begin{align*}
  a(v) & := -9.81 - 0.1 \cdot v \\
  i & := 0 \ldots 4060 \\
  \Delta t & := 0.001 \\
  t & := i \cdot \Delta t \\
  \begin{bmatrix}
    vA_0 \\
    yA_0 \\
    vB_0 \\
    yB_0
  \end{bmatrix} & :=
  \begin{bmatrix}
    0 \\
    100 \\
    30 \\
    0
  \end{bmatrix} \\
  \begin{bmatrix}
    vA_{i+1} \\
    yA_{i+1} \\
    vB_{i+1} \\
    yB_{i+1}
  \end{bmatrix} & :=
  \begin{bmatrix}
    vA_i + a(vA_i) \cdot \Delta t \\
    yA_i + vA_i \cdot \Delta t \\
    vB_i + a(vB_i) \cdot \Delta t \\
    yB_i + vB_i \cdot \Delta t
  \end{bmatrix}
\end{align*}
\]

The impact occurs 29.2 m above the original position of B (taken as the origin).

\[vB - vA_{4055} = 19.999\] The relative velocity at impact is 20 m/s
The Matlab code for this is:
```
function xdot=onept139(t,x);
xdot=[x(2);-9.81-0.1*x(2);x(4);-9.81-0.1*x(4)];
```
In the command window:
```
EDU>tspan=[0 4];
EDU>x0=[100;0;0;30];
EDU>[t,x]=ode45('onept139',tspan,x0);
EDU>plot(t,x(:,1),t,x(:,3))
```

1.140 Let $v_s$ denote the velocity of the swimmer starting from rest and $v_c = 0.1 + 0.1x_s$ denote the velocity of the river (opposite of $v_s$). Then $v_s/c = 1.5$ m/s, so that $v_s = v_c + 1.5 = -0.1 - 0.01x_s + 1.5$ or $\frac{dx_s}{dt} + 0.01x_s = 1.4$ which can be solved by use of an integrating factor for $x_s$. Here $p(t) = 0.01, f(t) = 1.4, \lambda(t) = e^{\int 0.01 dt} = e^{0.01t}$ so that $x_s(t) = e^{0.01t} [\int e^{0.01t} (1.4) + C_0] = 140 + C_0 e^{-0.01t}$. Since $x_s(0) = 0, C_0 = -140$ and $x_s(t) = 140(1 - e^{-0.01t}), x_s(60) = 140(1 - e^{-0.01 \times 60}) = 63.2$ m.

```
i:= 0..14000 Delta t := 0.01
v(x,t) := 1.4 - 0.01 x
x0 := 0
xi+1 := xi + v(xi,t) . Delta t
```

**FIGURE S1.140**
The Matlab code is:
\[x(1)=0; dt=0.001, t(1)=0\]
for n=1:14000
\[x(n+1)=x(n)+(1.4-0.01*x(n))*dt\]
\[t(n+1)=t(n)+dt\]
end
plot(t,x)

1.141 Since \(x_s(t) = 140(1 - e^{-0.6t})\) from problem 1.40, the swimmer will reach 100 m at time \(t\) that satisfies 100 = 140(1 - e^{-0.6t}) or \(t = \frac{-\ln(1/7)}{0.6} = 125\) s.

1.142 Given \(v_A(t) = \hat{j}\) m/s (constant), \(v_B(0) = 0\), \(a_B(t) = 2[\cos 30^{\circ}\hat{i} + \sin 30^{\circ}\hat{j}]\) n/s a constant. Also \(r_A(0) = 0\), \(r_B(0) = -4\hat{i} + \hat{j}\) m. From \(v_a(t) = 3\hat{j}\). \(r_A(t) = r_A(0) + 3\hat{j}\). Integration of \(a_B(t)\) yields: \(v_B(t) = v_B(0) + 1.73\hat{i} + t\hat{j}\) m/s = 1.73\hat{i} + t\hat{j}\. Integrating again yields \(r_B(t) = \frac{1.73}{2}t^2\hat{i} + \frac{1}{2}t^2\hat{j} + r_B(0) = (0.866t^2 - 4)\hat{i} + (0.5t^2 + 1)\hat{j}\) Thus \(r_{B/A}(t) = (3t + 4 - 0.866t^2)\hat{i} - (0.5t^2 + 1)\hat{j}\). For this to be zero, each component must be zero, which cannot be because the \(\hat{j}\) component is always positive, hence \(t\) has no real roots. The distance between the particles is \(|r_{B/A}| = \sqrt{(3t + 4 - 0.866t^2)^2 + (0.5t^2 + 1)^2}\). The minimum can be found from
\[
\frac{d}{dt}[(3t + 4 - 0.866t^2)^2 + (0.5t^2 + 1)^2] = 0
\]
\[2(3t + 4 - 0.866t^2)(3 - 1.73) + 2(0.5t^2 + 1)(t) = 0\]
which is cubic in \(t\) and has positive solutions. Then \(t = 0\) or \(r_{B/A} = 4\hat{i} = \hat{j}\) is when the two particles are closest. As time evolves they move apart (roots found using Mathcad).

1.143 Using \(y\) up along north and \(x\) to the right along east, the airplane has relative speed of \(v_{A/W} = 200(\cos \theta\hat{i} + \sin \theta\hat{j})\) mph where \(\theta\) is the unknown direction. The wind velocity is \(v_w = 50(\cos 45^{\circ}\hat{i} + \sin 45^{\circ}\hat{j}) = \frac{50}{\sqrt{2}}\hat{i} + \frac{50}{\sqrt{2}}\hat{j}\) = constant. We would like \(v_A = 200\hat{i}\). From the definition then
\[v_A = v_W + v_{A/W} = (50)[(0.707)\hat{i} + (0.707)\hat{j}] + 200(\cos \theta\hat{i} + \sin \theta)\hat{j}\]
which results in the two scalar equations
\[v_A = 35.4 + 200 \cos \theta\]
\[0 = 35.4 + 200 \sin \theta\]
Solving yields: \(\theta = -10.2^\circ\) and \(v_A = 232.2\) mph.
1.144 Note that the flanker travels 40 ft (20 ft/s × 2 s = 40 ft) before the quarter back throws the ball at \( t = 0 \). Let \( x_f(0) = 40 \hat{i} \) denote the initial position of the quarter back. For the ball
\[ a_b = -g \hat{k} \]
so that
\[ v_b = v \cos \theta \cos \beta \hat{i} + (-gt + v \sin \theta) \hat{j} + v \cos \theta \sin \beta \hat{k} \]
and
\[ x_b = \begin{bmatrix} v \cos \theta \cos \beta t + x_b(0) \hat{i} + \left[ -\frac{gt^2}{2} + v \sin \theta t + y_b(0) \right] \hat{j} + \left[ v \cos \theta \sin \beta t \right] \hat{k} \end{bmatrix} \]
where \( x_b(0) = -30 \text{ ft} \) (i.e. 30 yards behind the line of scrimmage) and \( y_b(0) = 6 \text{ ft} \). For the flanker \( x_f = 20 \hat{i} \), and integrating yields
\[ x_f(t) = (20t + x_f(0)) \hat{i} + 8 \hat{j} + 60 \hat{k} \]
Setting \( x_b = x_f \) yields the three equations
from \( \hat{i} \): \( 20t + 40 = v \cos \theta \cos \beta t - 30 \)
from \( \hat{j} \): \( 8 = -\frac{gt^2}{2} + v \sin \theta t + 6 \)
from \( \hat{k} \): \( 60 = v \cos \theta \sin \beta \).
Solving these three nonlinear equations in the three unknowns \( t, \beta, v \) for \( \theta = 30^\circ \) using a numerical procedure yields
\[ v = 69.719 \text{ ft/s}, \quad \beta = 28.152^\circ \text{ and } t = 2.106. \]
From the expression for \( x_f \) the yards gained on the play are
\[ \frac{(20)(2.106)+40}{3} = 27.375 \text{ yd} \]

1.145 Following sample 1.28, let \( v_s \) be the absolute velocity of the swimmer. With \( \hat{i} \) perpendicular to the shore and \( \hat{j} \) up stream, \( \mathbf{v}_s = v \hat{i} \), where \( v \) is the unknown absolute speed of the swimmer. Let \( \theta \) be the angle that \( \mathbf{v}_{s/w} \) (swimmer relative to the water) so that
\[ \mathbf{v}_{s/w} = 6 \cos \theta \hat{i} + 6 \sin \theta \hat{j} \text{ (ft/s)} \]
\[ \mathbf{v}_w = -5 \hat{j} \text{ ft/s, so that} \]
\[ \mathbf{v}_s = \mathbf{v}_w + \mathbf{v}_{s/w} \]
becomes
\[ \mathbf{v} = -5 \hat{j} + 6 \cos \theta \hat{i} + 6 \sin \theta \hat{j} \]
from \( \hat{i} \): \( v = 6 \cos \theta \)
from \( \hat{j} \): \( 0 = 6 \sin \theta - 5 \)
which is 2 equations in two unknowns: \( \theta \) and \( v \). Solving yields \( \theta = 56.4^\circ \), \( v = 3.3 \text{ ft/s} \). The time to travel 200 ft is \( t = \frac{200}{3.3} = 60.3 \text{ sec} \).
1.146 Solution:

We can form a block model of the United States as shown.

The direct flight from Seattle to Miami would be along a path South 21.8° of East. The wind direction for the 1200 miles was estimated to be South 45° East, during the next 600 miles the wind direction is to the East and during the last 1200 miles, the wind is North 45° East. The wind speed is estimated to be 50 mph.

The general relative motion equation for any segment of the flight is:

\[ \mathbf{v}_p = \mathbf{v}_w + \mathbf{v}_{p/w} \]

For the first leg of the flight, these vectors are:

\[ \mathbf{v}_p = \mathbf{v}_p \left[ \cos(21.8^\circ) \hat{i} - \sin(21.8^\circ) \hat{j} \right] \]
\[ \mathbf{v}_w = 50 \left[ \cos(45^\circ) \hat{i} - \sin(45^\circ) \hat{j} \right] \]
\[ \mathbf{v}_{p/w} = 300 \left[ \cos(\theta) \hat{i} + \sin(\theta) \hat{j} \right] \]

The scalar equations of relative motion are solved using the Given-Find function in Mathcad:

Make an initial guess for the velocity of the plane and the bearing of the plane

\[ \mathbf{v}_p := 300 \]
\[ \theta := 10^\circ - \text{deg} \]

Given

\[ \mathbf{v}_p \cdot \cos(21.8^\circ \text{deg}) = 50 \cdot \cos(45^\circ \text{deg}) + 300 \cdot \cos(\theta) \]
\[ -\mathbf{v}_p \cdot \sin(21.8^\circ \text{deg}) = -50 \cdot \sin(45^\circ \text{deg}) + 300 \cdot \sin(\theta) \]
1.147 Solution:

Wind Vel E S 85° E E N 30° E S 30° E

The solution will be made using the Given - Find function of Mathcad:

Leg 1.

Make an initial guess for the velocity of the plane and the bearing of the plane
\( v_p := 300 \)
\( \theta := 10 \cdot \text{deg} \)
Given
\( v_p \cdot \cos(21.8 \cdot \text{deg}) = 100 \cdot \cos(0 \cdot \text{deg}) + 300 \cdot \cos(\theta) \)
\( -v_p \cdot \sin(21.8 \cdot \text{deg}) = 100 \cdot \sin(0 \cdot \text{deg}) + 300 \cdot \sin(\theta) \)

Find \( \begin{bmatrix} v_p \\ \theta \end{bmatrix} = \begin{bmatrix} 390.541 \\ -0.505 \end{bmatrix} \)
\( \frac{0.505}{\text{deg}} = 28.934 \)
\( v_p = 391 \text{ mph} \quad \theta = -30 \cdot \text{deg} \)

Leg 2.

Make an initial guess for the velocity of the plane and the bearing of the plane
\( v_p := 300 \)
\( \theta := 10 \cdot \text{deg} \)
Given
\( v_p \cdot \cos(21.8 \cdot \text{deg}) = 100 \cdot \cos(85 \cdot \text{deg}) + 300 \cdot \cos(\theta) \)
\( -v_p \cdot \sin(21.8 \cdot \text{deg}) = -100 \cdot \sin(85 \cdot \text{deg}) + 300 \cdot \sin(\theta) \)

Find \( \begin{bmatrix} v_p \\ \theta \end{bmatrix} = \begin{bmatrix} 331.502 \\ -0.078 \end{bmatrix} \)
\( \frac{0.078}{\text{deg}} = 4.469 \)
1.148 Solution:

The flight is now to the West against the jet stream. The flight will be divided into three legs.

Leg 1.
Make an initial guess for the velocity of the plane and the bearing of the plane
\[ v_p := 200 \]
\[ \theta := 10 \text{-deg} \]

Given
\[ -v_p \cos(10 \text{-deg}) = 50 \cdot \cos(45 \text{-deg}) - 200 \cdot \cos(\theta) \]
\[ v_p \sin(10 \text{-deg}) = 50 \cdot \sin(45 \text{-deg}) - 200 \cdot \sin(\theta) \]

Find \( v_p, \theta = \begin{bmatrix} 167.082 \\ 0.032 \end{bmatrix} \)

\[ \frac{0.032}{\text{deg}} = 1.833 \]

The velocity of the plane and the course bearing are: \( v_p = 167 \text{ mph} \quad \theta = S 2^\circ W \)

Leg 2.
Make an initial guess for the velocity of the plane and the bearing of the plane
\[ v_p := 200 \]
\[ \theta := 10 \text{-deg} \]

Given
\[ -v_p \cos(10 \text{-deg}) = 50 \cdot \cos(0 \text{-deg}) - 200 \cdot \cos(\theta) \]
\[ v_p \sin(10 \text{-deg}) = 50 \cdot \sin(0 \text{-deg}) - 200 \cdot \sin(\theta) \]

Find \( v_p, \theta = \begin{bmatrix} 150.571 \\ -0.131 \end{bmatrix} \)

\[ \frac{0.131}{\text{deg}} = 7.506 \]
1.149 Solution:

![Diagram showing flight path from Chicago to Memphis]

<table>
<thead>
<tr>
<th>Wind Vel</th>
<th>750 mi</th>
<th>250 mi</th>
<th>500 mi</th>
<th>1000 mi</th>
<th>500 mi</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>S 85° E</td>
<td>E</td>
<td>N 30° E</td>
<td>S 30° E</td>
<td></td>
</tr>
</tbody>
</table>

The desired flight path from Chicago to Memphis is assumed to be S 80° E under a 100-mph wind N 30° E.

Make an initial guess for the velocity of the plane and the bearing of the plane:

$$v_p := 300$$

$$\theta := 80°$$

Given:

$$v_p \cos(80°) = 100 \cos(30°) + 300 \cos(\theta)$$

$$-v_p \sin(80°) = 100 \sin(30°) - 300 \sin(\theta)$$

Find $$\left( v_p, \theta \right) = \left[ \begin{array}{c} 250.701 \\ 1.715 \end{array} \right]$$

$$\frac{1.715}{\text{deg}} = 98.262$$

The velocity of the plane and the course bearing are: $$v_p = 251 \text{ mph}$$  $$\theta = S 82° W$$

1.150 From sample 1.29 the time to reach the opposite shore is $$t_f = \frac{2d}{v_{B/w} \cos \theta} = \frac{40}{\cos \theta}$$ sec. The expression for $$y(t)$$ at $$y(t_f) = 500$$ becomes

$$500 = 5 \left[ \left( \frac{10 \cos \theta}{200} \right)^2 - \frac{40^2}{3 \cos^3 \theta} - 2 \frac{10 \cos \theta}{200} \cdot \frac{40}{2 \cos^2 \theta} \right] + 10 \sin \theta \frac{40}{\cos \theta}$$

which is transcendental in $$\theta$$ and has solution (see numerical root finder) $$\theta = 56.43°$$.  

●
1.151 From sample 1.29 the angle is determined
\[ \theta = \sin^{-1} \left( \frac{2v_w}{3v_{B/w}} \right) = \sin \left( \frac{(2)(2)}{(3)(10)} \right) = 7.6^\circ \]
where \( v_w = 2 \text{ ft/s} \) is the velocity of the river.

1.152 From sample 1.30, the 3 equations determining \( v_s, \theta \) and \( \beta \) with \( v_{RT} = 0.7 \) become
\[ \theta + \beta = 90^\circ \]
\[ -v_s \cos \beta = -0.3 \cos \theta \]
\[ v_s \sin \beta = 0.7 - 0.3 \sin \theta \]
Solving using Mathcad yields: \( \theta = 24.8^\circ, \beta = 64.3^\circ \) and \( v_s = 0.632 \text{ m/s} \).

1.153 We are given: \( x_B = 2 \sin \pi t \hat{i} + 5t \hat{j} \),
\[ x_G = 4t \cos 30^\circ \hat{i} + 4t \sin 30^\circ \hat{j} \] so that differentiation yields
\[ v_B = 2\pi \cos \pi t \hat{i} + 5 \hat{j}, \quad v_G = 4 \cos 30^\circ \hat{i} + 4 \sin 30^\circ \hat{j} \] and
\[ a_B = -2\pi^2 \sin \pi t \hat{i}, \quad a_G = 0 \]
From eq. 1.126, \( v_{G/B} = v_G - v_B = (4 \cos 30^\circ - 2\pi \cos \pi t) \hat{i} + (4 \sin 30^\circ - 5) \hat{j} \)
and \( a_{G/B} = a_G - a_B = 2\pi^2 \sin \pi t \hat{i} \)
1.154 From the previous problem: 
\[ x_{G/A} = (4t \cos 30^\circ - 2 \sin \pi t)\hat{i} + (4t \sin 30^\circ - 5t)\hat{j} \]
so that the distance between the two as a function of time is 
\[ d(t) = |x(t)| = ((4t \cos 30^\circ - 2 \sin \pi t)^2 + (4t \sin 30^\circ - 5t)^2)^{1/2} \]
which is plotted below using Mathcad:

\[ \theta = 30 \cdot \text{deg} \quad t = 0, 0.1 \ldots 3 \]

\[ x_B(t) = \begin{bmatrix} 2 \cdot \sin (\pi \cdot t) \\ 5 \cdot t \\ 0 \end{bmatrix} \quad x_G(t) = \begin{bmatrix} 4 \cdot t \cdot \cos (\theta) \\ 4 \cdot t \cdot \sin (\theta) \\ 0 \end{bmatrix} \quad d(t) = |x_B(t) - x_G(t)| \]

**FIGURE S1.154**

1.155 Given \( x_B = 2 \sin \pi t\hat{i} + 5t\hat{j} \) and 
\[ x_G = 4t \cos 30^\circ \hat{i} + 4t \sin 30^\circ \left( \frac{\pi t}{2} \right) \hat{j}, \]
\[ x_{B/G} = (2 \sin \pi t - 4t \cos 30^\circ)\hat{i} + (5t - 4t \sin 30^\circ \sin \left( \frac{\pi t}{2} \right))\hat{j} \]
\[ v_{B/G} = (2 \pi \cos \pi t - 4 \cos 30^\circ)\hat{i} + (5 - 2\pi t \sin 30^\circ \cos \left( \frac{\pi t}{2} \right) - 4 \sin 30^\circ \sin \left( \frac{\pi t}{2} \right))\hat{j} \]
\[ a_{B/G} = -2\pi^2 \sin \pi t\hat{i} + (-4\pi \sin 30^\circ \cos \left( \frac{\pi t}{2} \right) + \pi^2 t \sin 30^\circ \sin \left( \frac{\pi t}{2} \right))\hat{j} \]
1.156 From \( \mathbf{x}_{B/0} \) of the previous problem:
\[
d(t) = |\mathbf{x}_{B/G}(t)| = [4\pi^4 \sin^2 \pi t + (1 - 4\pi \sin 30^\circ \cos \frac{\pi t}{2} + \pi t \sin 30^\circ \sin \frac{\pi t}{2})^2]^{1/2}\]
which is plotted in the figure using Mathcad.

\[
\theta := 30\cdot \text{deg} \quad t = 0, 0.1..3
\]

\[
\mathbf{x}_B(t) = \begin{bmatrix}
2 \cdot \sin(\pi t) \\
5 \cdot t \\
0
\end{bmatrix} \quad \mathbf{x}_G(t) = \begin{bmatrix}
4 \cdot t \cdot \cos(\theta) \\
4 \cdot t \cdot \sin(\theta) \cdot \sin\left(\frac{\pi t}{2}\right) \\
0
\end{bmatrix} \quad \mathbf{d}(t) := \mathbf{x}_B(t) - \mathbf{x}_G(t)
\]

\[\text{FIGURE S1.156}\]

The Matlab code is:

```matlab
syms t
th=pi/6;
xb=[2*sin(pi*t);5*t;0];xG=[4*t*cos(th);4*t*sin(th)*sin(pi*t/2);0];
d=sqrt(dot(xb,xG));
ezplot(d,[0,3])
```
From the given function
\[ a_A = \frac{d}{dt}(v_A) = \hat{i} + 1.224\hat{j} - 1.224\hat{k} - 0.2v_A \]
which is a vector differential equation of the form
\[ \ddot{v}_A + 0.2v_A = \hat{i} + 1.224\hat{j} - 1.224\hat{k} = \mathbf{b} \]

The solution of the homogeneous equation \( v_{Ah} = A e^{-0.2t} \) plus a particular solution: \( v_p = \frac{1}{0.2}(\hat{i} + 1.224\hat{j} - 1.224\hat{k}) \) or \( \mathbf{v}_A = A e^{-0.2t} + \frac{1}{0.2}(\hat{i} + 1.224\hat{j} - 1.224\hat{k}) \)
where \( A \) is evaluated from the listed condition \( v_A(0) = 0 \) which yields
\[ 0 = A + \frac{1}{0.2}(\hat{i} + 1.224\hat{j} - 1.224\hat{k}) \] or \( A = -\frac{1}{0.2}(1.224\hat{j} - 1.224\hat{k}) \)

Then \( \mathbf{v}_A = 5(1 - e^{-0.2t}) (\hat{i} + 1.224\hat{j} - 1.224\hat{k}) \text{ m/s} \)

Integration again yields
\[ \mathbf{x}_A - \mathbf{x}_A(0) = 5(t + 5e^{-0.2t} - 5)(\hat{i} + 1.224\hat{j} - 1.224\hat{k}) \]
or
\[ \mathbf{x}_A = 5(t + 5(e^{-0.2t} - 1))(\hat{i} + 1.224\hat{j} - 1.224\hat{k}) \]

Next consider \( \mathbf{a}_B = 3\hat{i} - 2\cos \pi t \hat{k} \text{ m/s} \). Integrating yields
\[ \mathbf{v}_B - \mathbf{v}_B(0) = \frac{3t^2}{2}\hat{i} - \frac{2\sin \pi t}{\pi} \hat{k} \text{ where } \mathbf{v}_B(0) = 0 \]

Integrating again yields
\[ \mathbf{x}_B - \mathbf{x}_B(0) = \frac{3t^3}{2}\hat{i} - \frac{2}{\pi}(1 - \cos \pi t) \hat{k} \text{ where } \mathbf{x}_B(0) = 0 \]

Now form
\[ \mathbf{v}_{B/A} = \left[ \frac{3t^2}{2} - 5(1 - e^{-0.2t}) \right] \hat{i} - 6.12(1 - e^{-0.2t}) \hat{j} \]
\[ + \left[ 6.12(1 - e^{-0.2t}) - \frac{2}{\pi} \sin \pi t \right] \hat{k} \]

and
\[ \mathbf{x}_{B/A} = \left[ \frac{t^3}{2} - 5(t + 5(e^{-0.2t} - 1)) \right] \hat{i} - \left[ 6.12(t + 5(e^{-0.2t} - 1)) \right] \hat{j} \]
\[ + \left[ \frac{1}{\pi^2}(\cos \pi t - 1) + 6.12(t + 5(e^{-0.2t} - 1)) \right] \hat{k} \]
The figure represents a numerical solution and plots of the magnitude of $x_{B/A}$ in Mathcad:

$$i = 0..3000 \quad \Delta t = 0.001 \quad t_i = i \cdot \Delta t$$

$$v_{x0} = 0 \quad x_0 := 0 \quad v_{y0} = 0 \quad y_0 = 0 \quad v_{z0} = 0 \quad z_0 := 0$$

$$ax(vx, t) = 1 - 0.2 \cdot vx - 3 \cdot t$$
$$ay(vy, t) = 2 \cdot 0.0612 - 0.2 \cdot vy$$
$$az(vz, t) = -0.621 \cdot 2 - 0.2 \cdot vz - 2 \cdot \cos(\pi \cdot t)$$

$$\begin{bmatrix}
v_{xi+1} \\
x_{i+1} \\
v_{yi+1} \\
y_{i+1} \\
v_{zi+1} \\
z_{i+1}
\end{bmatrix} = \begin{bmatrix}
v_{xi} - ax(v_{xi}, t_i) \cdot \Delta t \\
x_{i} - v_{xi} \cdot \Delta t \\
v_{yi} - ay(v_{yi}, t_i) \cdot \Delta t \\
y_{i} - v_{yi} \cdot \Delta t \\
v_{zi} - az(v_{zi}, t_i) \cdot \Delta t \\
z_{i} - v_{zi} \cdot \Delta t
\end{bmatrix}$$

$$v_i = \sqrt{(v_{xi})^2 - (v_{yi})^2 - (v_{zi})^2}$$
$$x_i = \sqrt{(x_i)^2 + (y_i)^2 + (z_i)^2}$$

FIGURE S1.157

The Matlab code ode45 can be used to solve this numerically. The plots of $x(t)$ and $r(t)$ can be obtained symbolically from:

```matlab
sys_t
x=[t^3/2-5*(exp(-0.2*b)-1));6.12*(t+5*(exp((-0.2*t)-1));(1/pi^2)*(cos(pi*t)-1)];
Nx=SVD(x);
ezplot(Nx(3),[0,3])
```
Define $d(t) = |\mathbf{x}_{B/A}|$ where $\mathbf{x}_{B/A}$ is defined as derived in Problem 1.157. Using Mathcad to plot yields figure S1.157.

Since $\mathbf{x}_A$ starts to move 1.5s after $\mathbf{x}_B$, shift the time of $\mathbf{x}_B$ ahead by 1.5s by replacing $t$ with $t + 1.5$ in the expression for $\mathbf{x}_B$ derived in Problem 1.157. This yields

$$\mathbf{x}_B = \left(\frac{(t+1.5)^3}{3}\right)\mathbf{i} - \frac{1}{\pi^2}(1 - \cos(\pi(t + 1.5)))\mathbf{k}$$

Then for $t \geq 4.5s$,

$$\mathbf{x}_{B/H} = \left(\frac{(t+1.5)^3}{2} - 5(t + 5(e^{-0.2t} - 1))\right)\mathbf{i} - 6.12[t + 5(e^{0.2t} - 1)]\mathbf{j} + \left\{6.12[t + 5(e^{-0.2t} - 1)] - \frac{2}{\pi^2}(1 - \cos(\pi t + 1.5))\right\}\mathbf{k}$$

and for $0 < t < 1.5$

$$\mathbf{x}_{B/A} = \left(\frac{(t+1.5)^3}{2} - \frac{2}{\pi^2}(1 - \cos(\pi(t + 1.5)))\right)\mathbf{k}.$$ 

The Mathcad solution:

$$i = 0..3000 \quad \Delta t = 0.001 \quad t_i = i \cdot \Delta t$$
$$v_{x0} = 0 \quad x_0 = 0 \quad v_{y0} = 0 \quad y_0 = 0 \quad v_{z0} = 0 \quad z_0 = 0$$
$$a_x(v_x, t) = \langle 1 - 0.2 \cdot v_x \rangle \cdot \Phi(t - 1.5) - 3 t^3$$
$$a_y(v_y, t) = \langle 2 \cdot 0.0612 - 0.2 \cdot v_y \rangle \cdot \Phi(t - 1.5)$$
$$a_z(v_z, t) = \langle -0.621 \cdot 2 - 0.2 \cdot v_z \rangle \cdot \Phi(t - 1.5) - 2 \cdot \cos(\pi \cdot t)$$

\[
\begin{align*}
&x_{i+1} \quad = x_i - v_x_i \cdot \Delta t \\
&y_{i+1} \quad = y_i - v_y_i \cdot \Delta t \\
&z_{i+1} \quad = z_i - v_z_i \cdot \Delta t \\
&v_x_i \quad = v_x_i = ax(v_x_i, t_i) \cdot \Delta t \\
&v_y_i \quad = v_y_i = ay(v_y_i, t_i) \cdot \Delta t \\
&v_z_i \quad = v_z_i = az(v_z_i, t_i) \cdot \Delta t \\
\end{align*}
\]

FIGURE S1.159
1.160 This is similar to sample problem 1.32. Using the coordinate system suggested in Sample 1.32, $y_B$ denotes the distance to the bracket and $x_A$ to the man. Then

\[
x_A + 2y_B = \ell \\
2v_B = -v_A \text{ or } v_A = -2v_B \\
\text{Since } v_B = 1 \text{ ft/s, } v_A = -2 \text{ ft/s (down)}
\]

1.161 Use the pulley as a reference point, and let $x_A$ be the distance from the pulley to $A$ and $x_B$ the distance from the pulley to $B$. Then

\[
x_A + x_B = \text{constant and differentiating yields} \\
x_A = -v_B, \text{ and } a_A = -a_B.
\]

1.162 Let $v_c$ be the cable velocity: $v_c = 0.5 \text{ m/s}$. Let $x_T$ denote the distance from the tree to the end of the truck. The length to the cable is $x_c + 2x_T = \ell$ so that

\[
v_c = -2v_T \text{ or } v_T = -\frac{1}{2}v_c = -0.25 \text{ m/s (right)}
\]

1.163 Let $x_A$ be the distance for the pulley to the man at $A$, $x_p$ be the distance between the top pulley and the bottom pulley. Let $x_B$ be the distance from the top pulley to $B$ and let the bracket for the bottom pulley be a height of $c$. Then

\[
x_A + 2x_p + x_B - c, \text{ but } x_p = (x_B - c) \\
\text{so that } x_A + 3x_B = \text{constant and } v_A = -3v_B \\
\text{with } x_B, x_A \text{ pointed down, } B \text{ traveling upwards yields} \\
v_B = -10 \text{ ft/s and } v_A = -3 \cdot (-10) = 30 \text{ ft/s (down)}
\]

1.164 Let $x_A$ denote the distance from the top of the pulley to block $A$, $y_B$ denote the distance from the top of the pulley down to the pulley holding $B$ and $y$ denote the distance from the top of the pulley down to mass $c$ (down position, to the right position). Then the length of the rope is

\[
y_c + 2y_B + x_A = \text{constant} \\
\text{differentiation yields} \\
v_A + 2v_B + v_C = 0 \\
or \\
v_c = -2v_B - 3 \text{ m/s and } a_c = -2a_A - 1 \text{ m/s}
\]
1.165 Let $h$ be the hypothesis of the triangle. Then from the right triangle

$$h^2 = 15^2 + x^2$$  \hspace{2cm} (1)

for any value of $x$. Also from length of rope

$$\ell = h + (15 - y) = 30$$  \hspace{2cm} (2)

Solve this for

$$h = y + 15$$  \hspace{2cm} (3)

Now substitute (3) into (1) to get:

$$(15 + a)^2 = 15^2 + x^2$$  \hspace{2cm} (4)

for any $x$ and $y$. Taking a time derivative yields

$$2(15 + y)\ddot{y} = 2x\dot{x}$$

Solving for $\dot{y}$ yields

$$\dot{y} = \frac{x\ddot{x}}{15 + y}$$

At $y = 10$

$$(15)^2 = 15^2 + x^2 \text{ or, } x = 20 \text{ m}$$

Then $\ddot{y} = \left(\frac{20}{25}\right) \left(0.5 \text{ m/s}^2\right) = 0.4 \text{ m/s}^2$, at $y = 10$ m.

1.166 Let $x_A$ extend from the fixed pulley above $B$ to the left of pulley $B$ to mass $A$ and let $y_B$ extend (positive) from the same pulley down to the mass at $B$. Let $d$ be the distance between the two fixed pulleys. Then the length of the rope is

$$\ell = 2y_B + x_A + (x_A - d) + (x_A - d)$$

or

$$\text{const.} = 2y_B + 3x_A.$$  \hspace{2cm} (5)

Differentiate to get:

$$\dot{y}_B = \frac{3}{2} \ddot{x}_A \text{ and } \ddot{y}_B = -\frac{3}{2} \ddot{x}_A.$$  \hspace{2cm} (6)

1.167 Use the top fixed pulleys as a reference and let $x_A$ denote the distance to mass $A$, $x_B$ the distance to mass $B$, $x_c$ the distance to mass $C$ and denote $x_p$ the distance to the pulley that is free to move. All distances are vertical with positive downward. There are two separate ropes. Let $\ell_1$ denote the from $A$ to the movable pulley so that

$$\ell_1 = x_A + x_B + (x_B - x_p)$$  \hspace{2cm} (1)
Let $\ell_2$ denote the length of the rope from mass $C$ up to ground so that

$$\ell_2 = x_c + 2x_p$$  \hspace{1cm} (2)

Differentiation of (2) yields $v_p = -v_c/2$ which is substituted into the derivative of (1) to yield

$$4v_B + v_c = -2v_A \text{ m/s}$$

\[\boxed{\text{1.168}}\]

Let the top pulley be the fixed frame of reference, and $x$ denote the distance from the reference down $(t)$ to the mass, and let $x_m$ be the distance from the reference down $(t)$ to a point on the rope connecting to the motor. Then

$$\ell = x_m + 2x$$

and differentiation yields that

$$v = \frac{1}{2}v_m$$

From section 1.7 $v_m = rw = (0.2)(9.55 \text{ rad/s} = 1.91 \text{ m/s}$ and $v = \frac{1}{2}v_m = 0.955 \text{ m/s}$

\[\boxed{\text{1.169}}\]

Let $y_B$ denote the distance from the pulley to the mass at $A$ (+ down), let $x_B$ denote the distance from the pulley to the collar along the shaft. The length of the rope is $\ell = d_a + y_B$. From the right triangle made by “$d$”, the bar the collar rides on, and the rope: $\ell_1^2$. Thus $\ell = d + y_B = y_B + \sqrt{d^2 + x_B^2}$. Differentiating yields $0 = \dot{y}_B + \frac{1}{2} \frac{2x_B \ddot{x}_B}{\sqrt{d^2 + x_B^2}}$.

\[\boxed{\text{1.170}}\]

Pick a fixed point to left of block $A$ as the reference and define $x_A$ and $x_c$ (both positive to the right) as the distance from the reference to block $A$ and $C$ respectively. Then define $y_c$ to be the distance down from the first pulley to $B$. The length of the rope $\ell$ is then $\ell = x_A + x_c + 2y_B$ and differentiation yields $v_A = -v_c - 2v_B$.

\[\boxed{\text{1.171}}\]

Solution:

$$\frac{ds}{dt} = 20 \text{ m/s} = \text{constant so } s = 20t \text{ and } \frac{ds^2}{dt^2} = 0$$

$$\theta(s) = 4\sin\left(\frac{s}{2000}\right) \text{ so that } \theta(t) = 4\sin(0.5t)$$

From equation 1.147 with $x_0 = y_0 = 0$

$$x(s) = \int_0^s \cos(\theta(\eta))d\eta \quad y(s) = \int_0^s \sin \theta(\eta)d\eta$$

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The Mathcad solution:

\[ s = 0, 10, \ldots, 6000 \]

\[ \theta(s) = 4 \cdot \sin \left( \frac{s}{2000} \right) \]

\[ x(s) := \int_{0}^{s} \cos(\theta(u)) \, du \quad y(s) := \int_{0}^{s} \sin(\theta(u)) \, du \]

\[ a_n(s) := 20^2 \frac{d}{ds} \theta(s) \]

\[ \text{Road as viewed from above} \]

From (1.146): \[ a_n = \left( \frac{ds}{dt} \right)^2 \frac{d\theta}{ds} = (20)^2 \frac{4}{200} \cos \left( \frac{s}{200} \right) = 800 \cos \left( \frac{s}{200} \right) \]

\[ y(s) \text{ and } |a_n(s)| \text{ are plotted in the figure.} \]

1.172 We are given that \( a(s) = 5 \) so that integrating \( vdv = 5ds \) yields \( v(s) = \sqrt{10s} \)
for zero initial condition. Also since \( \dot{\theta} = 1 + \frac{s^2}{2000^2}, \frac{d\theta}{ds} = \frac{2s}{1000^2}. \) But \( a_n(s) = v(s)^2 \frac{d\theta}{ds} = (10s)\left(\frac{2s}{2000^2}\right). \) Thus \[ |a| = \sqrt{a_n^2 + a_t^2} = \sqrt{(10s)^2\left(\frac{2s}{2000^2}\right)^2 + 25}. \] When \[ |a| = 8 \text{ the car spins out so that } s \text{ must satisfy} \]

\[ s^2 = (10s)^2\left(\frac{2s}{2000^2}\right)^2 + 25. \]

Solving for \( s \) yields: \[ s^2 = \frac{\sqrt{59}(2000)^2}{20} \text{ or } s = 1,118 \text{ m.} \]
1.173 The given constant speed means that \( \frac{ds}{dt} = 60 \text{ m/hour} \cdot \frac{\text{hour}}{3600 \text{ s}} = 88 \text{ ft/s} \). To determine the normal acceleration from equation (1.146) we need to calculate \((\frac{ds}{dt})^2 = 88^2\) and \(d\theta/ds\). From the given value of \(\theta\): \(\frac{d\theta}{ds} = \frac{1}{2000} \left( \frac{s}{2000} + e^{-s/1000} \right) = \frac{1}{2000} - \frac{1}{1000} e^{-s/1000} \) Thus \(a_n = \frac{88^2}{1000} \left[ 0.5 - e^{-s/1000} \right] = 7.744 \left[ \frac{1}{2} - e^{-s/1000} \right] \text{ ft/s}^2\). The Mathcad solution:

\[
\begin{align*}
\theta (s) &= \frac{s}{2000} - \frac{e^{1000}}{e^{1000}} \\
an(s) &= v^2 \frac{d}{ds} \theta (s)
\end{align*}
\]

![Magnitude of Normal Acceleration](figure_s1.173)

**FIGURE S1.173**

1.174 This requires the use of the 3-D formulation of equation 1.150. The component of \(\mathbf{a}\) normal to the above can be found by taking the dot product of the acceleration with the unit normal vector. To define the unit normal vector can be found from \(a_n = \mathbf{a} = \mathbf{a}_t = \mathbf{a} - \mathbf{a} \cdot \mathbf{t}\). First we need \(\frac{d\theta}{ds}\) and \(\frac{d\beta}{ds}\). From the given form: \(\frac{d\theta}{ds} = \frac{1}{1000} \left( 0.5 - e^{-s/1000} \right)\), \(\frac{d\beta}{ds} = \frac{\pi^2}{5000} \cos \left( \frac{\pi s}{5000} \right)\). The Mathcad solution:

\[
\begin{align*}
v &= 88 \quad s = 0, 10... 5000 \\
\theta (s) &= \frac{s}{2000} - \frac{e^{1000}}{e^{1000}} \\
an(s) &= v^2 \frac{d}{ds} \theta (s) = \pi \cdot \sin \left( \frac{\pi s}{5000} \right) \\
an(s) &= v^2 \left( \frac{d}{ds} \theta (s) \right) - \cos (\beta (s))^2 - \frac{d}{ds} \beta (s)
\end{align*}
\]

![normal acceleration in ft/s^2](figure_s1.174)

**FIGURE S1.174**
1.175 Solution:

\[
v = 88 \quad s = 0, 10 \ldots 5000 \\
v(s) = \sqrt{2 \cdot 1.5 \cdot s}
\]

\[
\theta(s) = \frac{s - \frac{s}{s}}{2000} \quad \beta(s) = \pi \cdot \sin \left( \frac{\pi \cdot s}{5000} \right)
\]

\[
an(s) = v(s)^2 \sqrt{\left( \frac{d}{ds} \theta(s) \right)^2 \cdot \cos(\beta(s))^2 - \left( \frac{d}{ds} \beta(s) \right)^2}
\]

\[
\rho(s) = \frac{1}{\sqrt{\left( \frac{d}{ds} \theta(s) \right)^2 \cdot \cos(\beta(s))^2 - \left( \frac{d}{ds} \beta(s) \right)^2}}
\]

**FIGURE S1.175**

1.176 Solution:

\[
v = 88 \quad s = 0, 10 \ldots 5000 \\
\theta(s) = \frac{s - \frac{s}{s}}{2000} \quad \beta(s) = \pi \cdot \sin \left( \frac{\pi \cdot s}{5000} \right)
\]

\[
\rho(s) = \frac{1}{\sqrt{\left( \frac{d}{ds} \theta(s) \right)^2 \cdot \cos(\beta(s))^2 - \left( \frac{d}{ds} \beta(s) \right)^2}}
\]

**FIGURE S1.176**
1.177 Solution:

\[
\begin{align*}
s &= 0, \, 1, \ldots, 50 \\
\theta(s) &= \frac{s}{100} \\
\beta(s) &= \frac{\pi}{2} \sin \left( \frac{\pi s}{25} \right) \\
v &= 10 \\
\end{align*}
\]

\[
an(s) = v^2 \sqrt{ \left( \frac{d}{ds} \theta(s) \right)^2 \cos(\beta(s))^2 + \left( \frac{d}{ds} \beta(s) \right)^2 }
\]

**FIGURE S1.177**

1.178 Solution:

\[
\begin{align*}
s &= 0, \, 1, \ldots, 100 \\
\theta(s) &= \frac{s}{100} \\
\beta(s) &= \frac{\pi}{2} \sin \left( \frac{\pi s}{25} \right) \\
v(s) &= 10 - 5 \cos \left( \frac{\pi s}{25} \right) \\
\end{align*}
\]

\[
an(s) = v(s)^2 \sqrt{ \left( \frac{d}{ds} \theta(s) \right)^2 \cos(\beta(s))^2 + \left( \frac{d}{ds} \beta(s) \right)^2 } \\
\text{at } s = \frac{d}{ds} v(s) \cdot v(s)
\]

\[
a(s) = \sqrt{ \left( \frac{d}{ds} \theta(s) \right)^2 - (\text{an}(s))^2 }
\]

**FIGURE S1.178**
2.1 Given: \( \mu_k = 0.1, W = 200 \text{ lb and } P = 50 \text{ lb} \). The free body diagram yields:

\[
N - mg - P \sin 30 = 0 \ (\text{sum of forces in } y\text{-direction}) \tag{1}
\]

\[
P \cos 30 - f = ma_x \ (\text{sum of forces in } x\text{-direction}) \tag{2}
\]

From (1) \( N = mg + P \sin 30 = 200 + 50(\frac{1}{2}) = 225 \text{ lb} \)

Friction force \( f = \mu_k N = 0.1(225) = 22.5 \text{ lb} \)

Therefore \( ma_x = 50(\frac{\sqrt{3}}{2}) - 22.5 = 20.80 \text{ lb} \)

\( \Rightarrow a_x = \frac{20.80}{200/32.2} = 3.34 \text{ ft/s}^2 \)

\( \Rightarrow v_x = 3.34t \) (assuming initially at rest)

\( \Rightarrow x = 1.67t^2 \) (assuming \( x(0) = 0 \))

Therefore \( t = \sqrt{\frac{100}{1.67}} = 7.73 \text{ sec} \)

![FIGURE S2.1](image)

2.2 Solution: \( \mu_s = 0.4, \ m = 150 \text{ kg}, \mu_k = 0.3 \)

\[
T - f - mg \sin \theta = ma_x
\]

\[
N - mg \cos \theta = ma_y = 0
\]

\( \Rightarrow N = mg \cos \theta = 150(9.81) \cos \theta = 150(9.81)\frac{100}{\sqrt{100^2 + 1^2}} = 1471 \text{ N} \)

\( \theta = \tan^{-1}(1/100) = 0.573^\circ \approx 0.01 \text{ rad} \)

![FIGURE S2.2](image)
\( f_{\text{max}} = \mu_s N = 0.4(1471) = 588.4 \text{ N} \)

Therefore \( T_{\text{max}} = f_{\text{max}} + mg \sin \theta = 588.4 + (150)(9.81) \frac{1}{\sqrt{100^2 + 1}} = 603 \text{ N} \)

\( f = \mu_k N = 0.3(1471) = 441 \text{ N} \)

\[ a_x = \frac{1}{m}(T - f - mg \sin \theta) = \frac{1}{150} \left[ 603 - 441 - 150(9.81) \frac{1}{\sqrt{100^2 + 1}} \right] \]

\( a_x = 0.98 \text{ m/s}^2 \)

Therefore \( v_x = 0.98t \text{ m/s assuming zero initial velocity.} \)

Therefore \( t = \frac{5}{0.98} = 5.1 \text{ sec} \)

2.3 Examining the maximum friction \( \mu_s = 0.2 \). This can produce an acceleration of

\( a_c = 0.2(32.2) = 6.44 \text{ ft/s}^2 \).

Therefore the box slips because the force of static friction \( ma_c \) is less than that applied by the truck

\( a_c = 0.15(32.2) = 4.83 \text{ ft/s}^2 \)

\( a_{c/T} = 4.83 - 10 = -5.17 \text{ ft/s}^2 \)

(We could now determine where the crate falls off if we knew length of tuck bed.)

Figure S2.3

2.4 Solution:

a) \( \sum F = ma \)

\[ 100 - 50 = \frac{100}{32.2} a \]

\( a = 16.1 \text{ ft/s}^2 \)

b) \( 100 - T = \frac{100}{32.2} a \)

\[ T - 50 = \frac{50}{32.2} a \]

Therefore \( 50 = \frac{150}{32.2} a \Rightarrow a = 10.73 \text{ ft/s}^2 \)

c) \( 150 - T = \frac{150}{32.2} a \)
\[ T - 100 = \frac{100}{32.2}a \]
\[ \Rightarrow 50 = \frac{250}{32.2}a \Rightarrow a = 6.44 \text{ ft/s}^2 \]

2.5 Solution:

\[ N - mg = ma, \text{ or} \]
\[ N = mg + ma \]

for: \( a = +2 \) : \( N_\uparrow = m(9.81 + 2) = 1181 \text{ N} \)
for: \( a = -2.5 \) : \( N_\downarrow = 100(9.81 - 2.5) = 731 \text{ N} \)

2.6 Solution:

\[-mg \sin 30^\circ + \mu mg \cos 30^\circ = ma \text{ along the conveyor. Or:} \]
\[-0.5g + 0.2598g = a, \text{ thus:} \]
\[ a = -0.24g. \]
\[ v = -0.24gt + v_0 \]
\[ x = -0.12gt^2 + 0.2t \]
0.12(9.81)t^2 - 0.2t - 10 = 0
\[ t = 3 \text{s} \]
Note that this slips immediately as \( \mu_s g \cos(30^\circ) < 0.5g \)

![FIGURE S2.6](image)

2.7 From S2.7a the constraint equation is
\[ 2(x_A - x_B) + (x_c - x_B) = \text{constant} \]
Thus \( 2x_A - 3x_B = \text{constant} \) and differentiation yields
\[ 2a_A = 3a_B \quad (1) \]
Now from S2.7b, Newton’s law in the horizontal direction yields
\[ 3T = (40)a_B \quad (2) \]
and from S2.7c, Newton’s law yields
\[ 100 - 2T = (20)a_A \quad (3) \]
This is 3 equations in 3 unknowns which yields
\[ a_B = 1.765 \text{ m/s}^2, \ a_A = 2.647 \text{ m/s}^2 \text{ and } T = 23.533 \text{ N} \]

![FIGURE S2.7](image)
2.8 Constraints to motion are from figure S2.8a are:

\[ x_c + x_p = \ell_1, \] therefore: \[ a_p = -a_c. \]

And \( x_A - x_p + x_B - x_p = \ell_2, \) hence:

\[ a_A + a_B - 2a_p = 0, \] and thus \( a_A + a_B + 2a_c = 0 \)

From the FBD’s of figure S2.8b:

\[ 25g - T_2 = 25a_A \]
\[ T_1 = 2T_2 \]
\[ 20g - T_2 = 20a_B \]
\[ 50g - 2T_2 = 50a_c. \]

But \( a_c = -\frac{1}{2}(a_A + a_B) \)

Subtract 2x the first equation from third to get

\[ 50(a_c - a_A) = 0, \] or \( a_c = a_A. \)

From the constraint \( a_B = -3a_A. \) Subtract 2nd from 1st

\[ 5g = 85a_A \rightarrow a_A = 0.577 \, \text{m/s}^2 \]
\[ T_2 = 231 \, \text{N} \quad a_B = -1.731 \, \text{m/s}^2 \]
\[ T_1 = 462 \, \text{N} \quad a_c = 0.577 \, \text{m/s}^2 \]
2.9 From figure S2.9a, the constraints are
\[ x_A + 2x_p = \ell_1, \] and
\[ (x_B - x_p) + x_B = \ell_2. \]

Differentiating yields \( a_A = -2a_p \) and \( 2a_B = a_p \) so that \( a_A = -4a_B \)

Next the FBD’s of Figure S2.9b yields:
\[ 100 \times 9.81 - T_1 = 100a_A \text{ from A} \]
\[ T_2 - 2T_1 = 0 \text{ (assumed massless) from P} \]
\[ (50)(9.81) - 2T_2 = 50a_B \text{ from B} \]

These three equations, plus the constraints \( a_A = -4a_B \) form a system of four equations and four unknowns which can be solved for \( a_A \). This yields
\[ a_A = 8.324 \text{ m/s}^2 \]
Integrating \( vdv = a_A dx \) from rest conditions yields
\[ \frac{v^2}{2} = (9.324)(0.5) \text{ or } v = 2.885 \text{ m/s} \]

2.10 From S2.10, \( (100)a = 100x \) or
\[ \ddot{x} = -50x \]
\[ vdv = -50dx \]
\[ \frac{v^2}{2} = -25(x^2 - 0.3^2) \]
\[ v = \sqrt{50(0.09 - x^2)} \]
\[ \frac{dx}{dt} = 7.07\sqrt{0.09 - x^2} \]
\[
\mathcal{F}_s \quad \text{FIGURE S2.10}
\]

Alternately from S2.10

\(-kx = m\ddot{x}\) where \(\dot{x}(0) = 0, x(0) = 0.3m\)

\(m = 2 \text{ kg}, \ k = 100 \text{ N/m}, \ x(0) = 0.3\)

\(\ddot{x} + 50x = 0\)

Integrating Factor

\(x = Ae^{at}, \ a^2 + 50 = 0, \ a = \pm 7.07i\)

\(x = A\sin(7.07t) + B\cos(7.07t)\)

\(x(0) = 0.3 = B \rightarrow B = 0.3\)

\(\dot{x}(0) = 0 = 7.07A \rightarrow A = 0\)

\(x = 0.3\cos(7.07t)\)

\[\int_{0.3}^{t} \frac{dx}{\sqrt{0.09-x^2}} = \int_{0}^{t} 7.07 dt\]

\(\sin^{-1}\left(\frac{x}{0.3}\right) \bigg|^{t}_{0.3} = 7.07t\)

\(\sin^{-1}\left(\frac{x}{0.3}\right) - \sin^{-1}(1) = 7.07t\)

\(\sin^{-1}\left(\frac{x}{0.3}\right) = (7.07t + \frac{\pi}{2})\)

\(x = 0.3\sin(7.07t + \frac{\pi}{2})\)

\(x = 0.3\cos(7.07t)\)

2.11 From the FBD of the system as a whole: \(P - 3mg\sin\alpha = 3ma\), so that

\(a = \frac{1}{3m}[P - 3mg\sin\alpha]\)

From the FBD for car C:

\(P - T_{BC} - mg\sin\alpha = ma = \frac{1}{3}[P - 3mg\sin\alpha].\)

Solving yields

\(T_{BC} = 2/3P.\)

Likewise a FBD of the system consisting of \(B\) and \(C\) taken together yields

\(T_{AB} = \frac{1}{3}P\)

\[\Box\]
2.12 From a FBD of the entire system as given in the top of Figure S2.12:

\[ 70 \sin 20^\circ - 70 \cos 20^\circ (0.2) = \frac{70}{32.2} a \]

\[ a = 4.96 \text{ ft/s}^2 \]

Now from the bottom of S2.12, the FBD yields

\[ 50 \sin 20^\circ - T - 0.2(50) \cos 20^\circ = \frac{50}{32.2} (4.96) \]

So \( T = 0! \)

This result is expected, as the motion is independent of mass and both blocks slides independently.

2.13 Sled will start to move when

a) \( 500(t^2 + 2t) = 0.5(200)(9.81) \quad t^2 + 2t - 1.962 = 0 \)

\[ t = -1 \pm \frac{\sqrt{4 + 4(1.962)}}{2} = -1 \pm \sqrt{2.962} \]

\[ t = 0.721 \text{ s or } t = -2.721 \]
b) From $t = 0.721$ to $6s$

$F(t) - 0.4(200)(9.81) = 200a$

$v = \frac{1}{200} \int_{0.721}^{6} [500(t^2 + 2t) - 0.4(200)9.81] dt$

$= [0.833t^3 + 2.5t^2 - 3.924t]_{0.721}^6$

$= 248 \text{ m/s (555 mi/hr)}$

$v(t) = 0.833t^3 + 2.5t^2 - 3.924t + 1.217$

$x(6) = [0.208t^4 + 0.833t^8 - 1.962t^2 + 1.217t]_{0.721}^6$

$x(6) = 386$ m

c) From $6s$ on:

$-0.4g = a$

$a = -3.924$

$v = -3.924t + 248$

time where $v = 0$

$t = 63.2s$

$x = -\frac{3.924}{2}t^2 + 248t$

$= 7837$ m

Total distance $= 386 + 7837 = 8223$ m or $8.2$ km (5.1 mi)
2.14 Solution: \( f_{\text{max}} = 0.3(90)(9.81) = 265 \) N. Therefore system slips.

a) Assume whole system slips as a whole. Then summing forces yields: \( 1200 - (0.25)(990)(9.91) = (90)(a) \). Solving yields \( a = 10.88 \text{ m/s}^2 \).

\[ \begin{array}{c}
\text{20} \\
\text{40} \\
\text{30} \\
\end{array} \quad \quad \begin{array}{c}
\text{1200N} \\
\text{90(9.81)} \\
\end{array} \quad \begin{array}{c}
f \\
\end{array} \]

\( a = 10.9 \text{ m/s}^2 \)

\[ \begin{array}{c}
\text{A + B + C} \\
\end{array} \quad \begin{array}{c}
\text{a=10.9 m/s}^2 \\
\end{array} \quad \begin{array}{c}
\text{1200} \\
\end{array} \]

\( f = 0.3(60)(9.81) = 176.58 \) 

A+B slips

**FIGURE S2.14a**

b) Assume slip between each surface

\[ \begin{array}{c}
\text{A+B} \\
\text{60} \\
\end{array} \quad \begin{array}{c}
f=60a \\
\text{a=2.9 m/s}^2 \\
\end{array} \]

\( f = 0.3(60)(9.81) = 176.58 \) 

A+B slips

**FIGURE S2.14b**

\( 0.25(20)(9.81) = 20a_A \) from FBD of the 20kg block. Thus \( a_A = 2.45 \text{ m/s}^2 \)

\( 0.25(60)(9.81) - 0.25(20)(9.81) = 40a_B \) from the FBD of the 40kg block thus \( a_B = 2.45 \text{ m/s}^2 \)

Therefore \( A \) and \( B \) the top two blocks move together (they have the same acceleration)
c) Next $C$ slips out from under $A$ and $B$ (the FBD is S2.14c). Then
$$0.25(60)(9.81) = 60a_{AB}.$$ Thus
$$a_{AB} = 2.45 \text{ m/s}^2.$$ From the FBD on the bottom of Figure S2.14c:
$$1200 - 0.25(60)(9.81) - 0.25(90)(9.81) = 30a_c$$ Thus the acceleration of the bottom block is
$$a_c = 27.74 \text{ m/s}^2$$

![FBD of S2.14c]

2.15 Will the system slip?

![FBD of S2.15a]

Is $400 \geq 0.3(90)(9.81) = 265 \text{ N}$?
Yes, so the system slips.
From the FBD of $A$ and $B$:

$$400 - 0.25(90)9.81 = 90a_T$$

$$a_T = 1.99 \text{ m/s}^2$$

$$f_{\text{max}} = 176.6$$ which implies $$a = 2.9 \text{ m/s}^2$$

So $A$ and $B$ must move with $C$.

2.16 Solution:

![Diagram of forces](https://via.placeholder.com/150)

**FIGURE S2.16a**

Assume no slip and determine the required value of $\mu_s$. From FBD of $A$:

$$N_1 + \mu N_2 \cos 30^\circ - N_2 \sin 30^\circ = 0$$  \hfill (1)

$$\mu N_1 + \mu N_2 \sin 30^\circ + N_2 \cos 30^\circ - 50g = 0$$  \hfill (2)

From the FBD of $B$:

$$-\mu N_2 \cos 30^\circ + N_2 \sin 30^\circ - \mu N_3 = 0$$  \hfill (3)

$$-\mu N_2 \sin 30^\circ - N_2 \cos 30^\circ + N_3 - 10g = 0$$  \hfill (4)

This is a system of four nonlinear equations in the four unknowns $\mu_s$, $N_1$, $N_2$, and $N_3$. Solved by Mathcad Eqs. 1-4 required $\mu_s = 0.242$.

Since the $\mu$ required for equilibrium is larger than the 0.2 given the system will slip. Knowing it will slip, the equations of motion are (with zero acceleration into the walls)

$$N_1 + \mu_k N_2 \cos 30^\circ - N_2 \sin 30^\circ = 0$$

$$\mu_k N_1 + \mu_k N_2 \sin 30^\circ + N_2 \cos 30^\circ - 50g = 50a_A$$

$$-\mu_k N_2 \cos 30^\circ + N_2 \sin 30^\circ - \mu_k N_3 = 10a_B$$

$$-\mu_k N_2 \sin 30^\circ - N_2 \cos 30^\circ + N_3 - 10g = 0$$

$\mu_k = 0.15$, this is a system of 5 unknowns $N_1$, $N_2$, $N_3$, $a_A$, $a_B$ and only 4 equations. Hence we need a constraint of the motion. The acceleration of the
blocks in the normal direction at the contacting surface is the same (see S2.16b). Thus \(-a_A \cos 30^\circ = a_B \sin 30^\circ\) which provides the 5th equation. Solving these 5 nonlinear equations in the 5 unknowns yields the desired acceleration (using a computer code such as the MATLAB file at the end of this solution):

\[
a_A = -3.401 \text{ m/s}^2
\]
\[
a_B = 5.891 \text{ m/s}^2
\]

From \(d = \frac{1}{2}a t^2\), the time for box A to move 0.1 m is

\[
t = \sqrt{\frac{(2)(0.1)}{3.401}} = 0.242 \text{ s}
\]
During this time box B moves

\[
x_B(0.242) = \frac{5.891}{2}(0.242)^2 = 0.173 \text{ m}
\]

This must be added to the distance the block B slides to the right due to its separation velocity. The velocity of separation is:

\[
v_B = 5.891 \cdot 0.242 = 1.426 \text{ m/s}
\]

The deceleration of the block is \(-0.15 \cdot 9.81 = -1.472\) and the time when the block stops is:

\[
t = \frac{1.426}{1.472} = 0.969 \text{ s}
\]

The total distance traveled is:

\[
x_B = -0.075 \cdot 9.81(0.969)^2 + 1.426 \cdot 0.969 + 0.173 = 0.864 \text{ m}
\]
The MATLAB solution:

**Main program to solve the first nonlinear equation**('non1.m')
% Main m-file to solve the nonlinear equation
% Initial guess values for the solution
% x(1)=N1; x(2)=N2; x(3)=N3; x(4)=Mu_s;
g=9.81;
x0=[ 10*g 6*g 10*g 0.4];
% solve the nonlinear equation with given initial guess
F=fsolve('neqn1',x0);
% display solutions
disp('               ')
disp(sprintf('   N1= %6.4f',F(1)))
disp(sprintf('   N2= %6.4f',F(2)))
disp(sprintf('   N3= %6.4f',F(3)))
disp(sprintf('   Mu_s= %6.4f',F(4)))

**Function program to solve the first nonlinear equation**('neqn1.m')
function F=neqn1(x)
% Function m-file to define the nonlinear equation
deg2rad=pi./180; g=9.81; % constants required for the calculation
N1=x(1); N2=x(2); N3=x(3); Mu_s=x(4); % assign the values
F1= N1 + Mu_s * N2 * cos(30*deg2rad) - N2 * sin(30*deg2rad) ;
F2= Mu_s * N1 + Mu_s * N2 * sin(30*deg2rad) + N2 * cos(30*deg2rad) - 50*g ;
F3= -Mu_s * N2 * cos(30*deg2rad) + N2 * sin(30*deg2rad) - N3 * Mu_s ;
F4= -Mu_s * N2 * sin(30*deg2rad) - N2 * cos(30*deg2rad) + N3 - 10*g ;
F=[F1 F2 F3 F4]';

**Main program to solve the second nonlinear equation**('non2.m')
% Main m-file to solve the nonlinear equation
% Initial guess values for the solution
% x(1)=a_A; x(2)=a_B; x(3)=N1; x(4)=N2; x(5)=N3;
g=9.81;
x0=[ -0.3 0.4 10*g 6*g 10*g ];
% solve the nonlinear equation with given initial guess
F=fsolve('neqn2',x0);
% display solutions
disp('               ')
disp(sprintf('   a_A= %6.4f',F(1)))
disp(sprintf('   a_B= %6.4f',F(2)))
disp(sprintf('   N1= %6.4f',F(3)))
disp(sprintf('   N2= %6.4f',F(4)))
disp(sprintf('   N3= %6.4f',F(5)))

**Function program to solve the first nonlinear equation**('neqn2.m')
function F=neqn2(x)
% Function m-file to define the nonlinear equation
deg2rad=pi./180; g=9.81; Mu_k=0.15; % constants required for the calculation
a_A=x(1); a_B=x(2); N1=x(3); N2=x(4); N3=x(5); % assign the values
F1= N1 + Mu_k * N2 * cos(30*deg2rad) - N2 * sin(30*deg2rad) ;
F2= Mu_k * N1 + Mu_k * N2 * sin(30*deg2rad) + N2 * cos(30*deg2rad) - 50*g - 50 * a_A ;
F3= -Mu_k * N2 * cos(30*deg2rad) + N2 * sin(30*deg2rad) - N3 * Mu_k - 10 * a_B ;
F4= -Mu_k * N2 * sin(30*deg2rad) - N2 * cos(30*deg2rad) + N3 - 10*g ;
F5= a_A * cos(30*deg2rad) + a_B*sin(30*deg2rad);
F=[F1 F2 F3 F4 F5]';
2.17 The geometry of the sketched spring in Fig. S2.17 yields
\[ \ell^2 = \ell_0^2 + x^2 \]
\[ (\ell_0 + \Delta)^2 = \ell_0^2 + x^2 \]
\[ \ell_0^2 + 2\ell_0\Delta + \Delta^2 = \ell_0^2 + x^2 \]
where \( \Delta \) is the spring elongation.

\[ \Delta = \frac{-2\ell_0 \pm \sqrt{4\ell_0^2 + 4x^2}}{2} = -\ell_0 \pm \sqrt{\ell_0^2 + x^2} \]
Using the positive root, the magnitude of the spring force is
\[ |F_s| = k \left[ \sqrt{\ell_0^2 + x^2} - \ell_0 \right] \]

From the FBD:
\[ mg - k \left[ \sqrt{\ell_0^2 + x^2} - \ell_0 \right] \frac{x}{\sqrt{\ell_0^2 + x^2}} = ma \]
Thus
\[ \frac{d^2x}{dt^2} = g - \frac{k}{m} \left[ \sqrt{\ell_0^2 + x^2} - \ell_0 \right] \frac{x}{\sqrt{\ell_0^2 + x^2}} \]
\[ a(x) = g - \frac{kx}{m} \left( 1 - \frac{\ell_0}{\sqrt{\ell_0^2 + x^2}} \right) \]

2.18 Solution:

From the figure
\[ (\ell + \Delta)^2 = x^2 + \ell^2 \]
\[ \ell^2 + 2\ell\Delta + \Delta^2 = x^2 + \ell^2 \]
\[ \Delta^2 + 2\ell\Delta - x^2 = 0 \]
\[ \Delta = \frac{-2\ell \pm \sqrt{4\ell^2 + 4x^2}}{2} = \sqrt{\ell^2 + x^2} - \ell \] (could write this directly)
\[ mg - F_s \sin \theta - \mu_k N = m\ddot{x}, \quad \sin \theta = \frac{x}{\sqrt{\ell^2 + x^2}} \]
\[ N - F_s \cos \theta = 0, \quad \cos \theta = \frac{\ell}{\sqrt{\ell^2 + x^2}} \]
FIGURE S2.18

\[ N = k \left[ \sqrt{\ell^2 + x^2} - \ell \right] \frac{\ell}{\sqrt{\ell^2 + x^2}}, \quad F_s = k \Delta \]

\[ N = k \ell \left[ 1 - \frac{\ell}{\sqrt{\ell^2 + x^2}} \right] \]

\[ \ddot{x} = g - \frac{k}{m} \left[ x \left( 1 - \frac{\ell}{\sqrt{\ell^2 + x^2}} \right) + \mu_k \frac{\dot{x}}{|\dot{x}|} \ell \left( 1 - \frac{\ell}{\sqrt{\ell^2 + x^2}} \right) \right] \]

\[ \ddot{x} = g - \frac{k}{m} (x + \mu_k \frac{\dot{x}}{|\dot{x}|} \ell)(1 - \frac{\ell}{\sqrt{\ell^2 + x^2}}) \]

which would require a numerical solution.
2.19 Solution:

\[ l := 0.3 \quad k := 300 \quad m := 2 \quad g := 9.81 \]
\[ i := 0..2000 \quad \Delta t := 0.001 \]
\[ \begin{bmatrix} v_0 \\ x_0 \end{bmatrix} := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ a(x) := g - \frac{k}{m} \frac{x}{\sqrt{l^2 + x^2} - l} \]
\[ \begin{bmatrix} v_{i+1} \\ x_{i+1} \end{bmatrix} := \begin{bmatrix} v_i + a(x_i) \Delta t \\ x_i + v_i \Delta t \end{bmatrix} \]

2.20 First note that \( 30g \geq 0.5(20)g \), so it does in fact slip. The constraint is that

\[ L = x_A + x_B + \text{const} \]
\[ \Rightarrow 0 = \ddot{x}_A + \ddot{x}_B \]
\[ \Rightarrow 0 = a_A + a_B \text{ so that } a_A = -a_B. \] (1)

From the FBD of \((B)\):
\[ 30(9.8) - T = 30a_B \] (2)

From the FBD of \(A\): \( \sum F_x = ma_A \) becomes:
\[ T - \mu_k N = 20a_A \]
\[ T - \mu_k mg = 20a_A \]
\[ T - 0.4(20)(9.81) = (20)a_A \] (3)

Adding (2) and (3), using (1) yields

\[ (30)(9.81) - (0.4)(20)(9.81) = 50a_A \]
So that \( a_A = 4.316 \text{ m/s}^2 \). Then \( d = \frac{a_A}{2} t^2 \) or
\[
t = \sqrt{\frac{1}{4.326}} = 0.481 \text{ s}.
\]

2.21 First determine the unit normal vector to the surface of the ice, \( n \).

Then compute the component of the weight vector along \( n \), denoted \( N \). The force parallel to the ice will then be \( S = W - N \), where \( W \) is the weight. Once the unit normal along \( S \) is determined, the motion can be treated as rectilinear along the face of the ice. The following Mathcad code completes the solution.
\[
T := \begin{bmatrix}
-\cos(45\text{ deg}) \\
0 \\
\sin(45\text{ deg})
\end{bmatrix},
t := \begin{bmatrix}
0 \\
-\cos(30\text{ deg}) \\
\sin(30\text{ deg})
\end{bmatrix},
W := \begin{bmatrix}
0 \\
0 \\
-130
\end{bmatrix}
\]

\[
n := \frac{T \times t}{|T \times t|}
\]

\[
n = \begin{bmatrix}
0.655 \\
0.378 \\
0.655
\end{bmatrix}
\]

Let us find the component of the weight that acts normal to the face of the ice.

\[
N := (W \cdot n) \cdot n
\]

\[
N = \begin{bmatrix}
-55.714 \\
-32.167 \\
-55.714
\end{bmatrix}
\]

The force parallel to the ice face causing her to slip is \( S \)

\[
S := W - N
\]

\[
S = \begin{bmatrix}
55.714 \\
32.167 \\
-74.286
\end{bmatrix}
\]

We can first see if she slips by comparing \(|S|\) to \(\mu_k |N|\)

\(|S| = 98.271, 1.0 \cdot |N| = 85.105\)

Therefore she slips.

The unit vector in the direction of the slip is \( s \)

\[
s := \frac{S}{|S|}
\]

We can now treat the motion as rectilinear along the ice face in the direction.

\[
a := \frac{32.2}{130} \left( |S| - 0.8 \cdot |N| \right)
\]

\[
a = 7.477 \text{ ft/s}^2
\]

The distances she slides in the \( s \) direction is:

\[
d = -100(t \cdot s) = 66.144 \text{ ft}
\]

Now her velocity after sliding 66.144 ft can be determined.

\[
v \frac{dv}{dx} = a, v = \sqrt{2ad}, v = 31.45 \text{ ft/s}. \text{ Yes, she will be injured.}
\]
2.22 Constraint: $x_A + x_B = \ell$ therefore $a_A = -a_B$

From the FBD of $A$:
\[20g \sin 30 + 0.2(20g) \cos 30 - T = 20a_A\]
From the FBD of $B$:
\[60g \sin 30 - 0.2(20g) \cos 30 - 0.1(80g) \cos 30 - T = 60a_B\]
Thus
\[132.1 - T = 20a_A\]
\[192.4 - T = 60a_B\]
Which along with the constraint equation is a system of 3 equations in 3 unknowns. Solving yields
\[a_B = 0.753 \text{ m/s}^2\]
\[a_A = -0.753 \text{ m/s}^2\]
\[a_{A/B} = a_A - a_B = -1.506 \text{ m/s}^2\]
2.23 The rope length yields: $2x_A + y_B = \ell$, so that $2a_A + a_B = 0$

![FIGURE S2.23](image)

A FBD of $A$ yields:

$$150 - 2T = 8a_A \quad (1)$$

A FBD of $B$ yields:

$$3 \cdot (9.81) - T = 3a_B \quad (2)$$

Using the constraint, the 1st equation becomes

$$150 - 2T = -4a_B$$

Multiply (2) by 2 yields

$$58.86 - 2T = 6a_B$$

Subtracting these last two equations yields

$$-10a_B = 91.14$$

so $a_B = -9.114 \text{ m/s}$, and $a_A = 4.557 \text{ m/s}^2$

Also from (2): $T = 3(9.81) - 3a_B = 56.8 \text{ N}$

Solving

$v_B = a_B t$

for $t$ yields:

$$t = \frac{2.5}{4.557} = 0.549 \text{ s}$$

2.24 The constraint here is that $x_A + x_B = \text{constant}$, so that $a_A = -a_B$. Assume that the system moves to the left.

From FBD of mass $B$

for $y$ direction: $N_B - m_B g \cos \beta = 0$, so $N_B = m_B g \cos \beta$ and $f_B = \mu_k N_B = \mu_k m_B g \cos \beta$

from $x$ direction: $-T + m_B g \sin \beta - f_B = m_B a_B$

Thus we have: $a_B = g(\sin \beta - \mu_k \cos \beta) - \frac{T}{m_B} \quad (1)$
From the FBD of mass $A$
from the $y$-direction: $N_A - m_A g \cos \alpha = 0$ or $N_A = m_A g \cos \alpha$
from the $x$-direction: $-T + m_A g \sin \alpha - \mu_k m_A g \cos \alpha = m_A a_A$
Thus $a_A = \ddot{x}_A = g(\sin \alpha - \mu_k \cos \alpha) - \frac{T}{m_A}$. \hfill (2)

Now use the constraint $a_A + a_B = 0$ along with equations (1) and (2) to eliminate
the tension $T$. This yields
\[
g(\sin \alpha - \mu_k \cos \alpha) - \frac{T}{m_A} + g(\sin \beta - \mu_k \cos \beta) - \frac{T}{m_B} = 0
\]
Solving yields $T = g \left(\frac{m_A m_B}{m_A + m_B}\right) (\sin \alpha + \sin \beta - \mu_k (\cos \alpha + \cos \beta))$

Thus the value of constant acceleration from (2) is:
\[
\ddot{x}_A = g(\sin \alpha - \mu_k \cos \alpha) - g \left(\frac{m_A m_B}{m_A + m_B}\right) [\sin \alpha + \sin \beta - \mu_k (\cos \alpha + \cos \beta)]
\]

2.25 Solution:

From the FBD, the sum of forces in the $x$ direction yields:
\[
N - mg = ma \sin \theta
\]
The force in the $y$ direction is:

$$\mu_s N = ma \cos \theta$$

Thus

$$\mu_s (mg + ma \sin \theta) = ma \cos \theta$$

or

$$\mu_s g = a(\cos \theta - \mu_s \sin \theta)$$

or

$$a = \frac{\mu_s g}{\cos \theta - \mu_s \sin \theta}.$$ 

2.26 Solution:

From the FBD, the force sum in the $x$ direction is:

$$\mu_s N = ma \cos \theta$$

The force sum in the $y$ direction is:

$$mg - N = ma \sin \theta$$

Thus:

$$N = m(g - a \sin \theta)$$

and

$$\mu_s m(g - a \sin \theta) = ma \cos \theta$$

$$\mu_s g = a(\cos \theta + \mu_s \sin \theta)$$

$$a = \frac{\mu_s g}{\cos \theta + \mu_s \sin \theta}.$$
2.27 From the top drawing, the constraints are

\[ 2y_A + x_p = \ell_1 \] and \[ x_B + 2(x_B - x_p) = \ell_2. \]
Thus \( 3a_B + 4a_A = 0 \) \hspace{1cm} (1)

From the FBD of the pulley, \( T_1 = 2T_2. \) Thus the FBD of \( A \) yields:
\[ 20(9.81) - 4T_2 = 20a_A \] \hspace{1cm} (2)

The FBD of \( B \) yields:
\[-3T_2 = 10a_B \rightarrow -3T_2 = -10\frac{4}{3}a_A \text{ so } T_2 = \frac{40}{9}a_A \] \hspace{1cm} (3)

Substitution of (3) into (2) yields:
\[ 20(9.81) - 4(\frac{40}{9})a_A = 20a_A \]
or
\[ (180 + 160)a_A = 9(20)9.81 \]
and
\[ a_A = 5.194 \text{ m/s}^2. \]

\[ x_A(t) = \frac{a_A}{2}t^2 \] thus
\[ 1.5 = \frac{5.194}{2}t^2, \ T_2 = 23.1 \text{ N}, \ t = 0.76 \text{ s}, \ T_1 = 46.2 \text{ N}. \]

2.28 From the drawing, the constraints are:
\[ x_A + x_B + (x_B - x_p) = \ell_2 \] and \[ 2x_p + x_c = \ell_1 \] combining yields
\[ x_A + 2x_B + \frac{x_c}{2} = \text{const} \]

Thus: \( 2a_A + 4a_B + a_c = 0. \) Let \( T_1 \) be the tension in cable 1 and \( T_2 = T_1/2 \) be the tension in cable 2.
Now since $a = \frac{1}{2}at^2$, $t = \sqrt{\frac{2(a)}{a}} = \sqrt{\frac{(2)(4)}{4.5}} = 1.33$ s.

From the FBD of A: $10g - T_1 = 10a_A$

From the FBD of B: $80g - 2T_1 = 80a_B$

From the FBD of C: $20g - \frac{T_1}{2} = 20a_c$

Rearranging these 4 equations in 4 unknowns yields:

- $10a_A + T_1 = 10g$
- $80a_B + 2T_1 = 80g$
- $20a_c + \frac{T_1}{2} = 20g$
- $2a_A + 4a_B + a_c = 0$

In matrix form these become:

$$\begin{bmatrix} a_A \\ a_B \\ a_C \\ T_1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 & 1 \\ 0 & 80 & 0 & 2 \\ 0 & 0 & 20 & 0.5 \\ 2 & 4 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 10 \cdot 9.81 \\ 80 \cdot 9.81 \\ 20 \cdot 9.81 \\ 0 \end{bmatrix}$$

which can be solved using a code to yield:

- $a_A = -11.319$, $a_B = 4.528$, $a_C = 4.528$,
- $T_1 = 211.292$ and $T_2 = \frac{T_1}{2} = 105.646$

Now since $a = \frac{1}{2}at^2$, $t = \sqrt{\frac{2a}{a}} = \sqrt{\frac{(2)(4)}{4.5}} = 1.33$ s.

2.29 Solution:

Does system slip? $100 > 30(9.81)(0.2) = 58.9$ so it slips.

From the FBD of the system as a whole:

$100 - 0.15(30)(9.81) = 30a_T$

Thus

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\(a_T = 1.862 \text{ m/s}^2\)

From the FBD of \(B\) \(0.2(20)9.81 = 20a_B\) or \(a_B = 1.962\) which is larger than \(a_T\) so there is enough friction for the system to move as one.

2.30 Solution:

\[
\begin{align*}
(0.15)(20)g &= 20a_B \\
\text{From A:} \\
100 - (0.15)(20)g - (0.15)(30)g &= 10a_A
\end{align*}
\]

So \(a_A = 2.643 \text{ m/s}^2\)

Block B slips.

2.31 Solution:

Note that \(\mathbf{a}_A = a \cos \theta \hat{i} + a \sin \theta \hat{j}\) and \(\mathbf{a}_B = \ddot{x}_B \hat{i} + \ddot{y}_B \hat{j}\)

so that

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\[ \mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A = (\ddot{x}_B - a \cos \theta) \hat{i} + (\ddot{y}_B - a \sin \theta) \hat{j} \]

Since \( a_{B/A} \) cannot have a component in the \( \hat{j} \) direction we have the constraint that
\[ \ddot{y}_B = a \sin \theta \]  \hspace{1cm} (1)

From the FBD of \( B \) we have
\( (x \text{ direction}) \) \( 0 = m_B \ddot{x}_B \) so that \( \ddot{x}_B = 0 \)
\( (y \text{ direction}) \) \( N_B - m_B g = m_B \ddot{y}_B \)  \hspace{1cm} (2)

From the FBD of \( A \) we have
\( (x \text{ direction}) \) \( -N_A \sin \theta = m_A \ddot{x}_A = m_A a \cos \theta \)  \hspace{1cm} (3)
\( (y \text{ direction}) \) \( -N_B - m_A g + N_A \cos \theta = m_A a \sin \theta \)  \hspace{1cm} (4)

This yields a system of 4 equations in the 4 unknowns \( y_B, a, N_A \) and \( N_B \) which can be written in matrix form as
\[
\begin{bmatrix}
1 & -\sin \theta & 0 & 0 \\
30 & 0 & 0 & -1 \\
0 & 50 \cos \theta & \sin \theta & 0 \\
0 & 50 \sin \theta & -\cos \theta & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{y}_B \\
a \\
N_A \\
N_B
\end{bmatrix}
= \begin{bmatrix}
0 \\
-(30)(9.81) \\
-(30)(9.81) \\
0
\end{bmatrix}
\]

This has solution
\[ a = -6.824 \text{ m/s}^2, \quad \ddot{y}_B = -3.411 \text{ m/s}^2, \quad N_A = 591.0 \text{ N}, \quad N_B = 191.9 \text{ N}. \]

The relative acceleration is
\[ \mathbf{a}_{B/A} = (\ddot{x}_B - a \cos \theta) \hat{i} \]

which has magnitude
\[ 0 + (6.824)(.866) = 5.91 \text{ m/s}^2. \]

Thus
\[ t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{2(1)}{5.91}} = 0.184 \text{ s} \]
2.32 Solution:

The spring force is
\[ F_s = k\Delta. \]

From the geometry
\[ \Delta = \sqrt{l^2 + x^2} - l. \]

Summing forces in the \( x \) direction (see S2.32) yields
\[ mg \cos \alpha - k(\sqrt{l^2 + x^2} - l)\frac{x}{\sqrt{l^2 + x^2}} - \mu_k N \frac{\dot{x}}{|\dot{x}|} = m\ddot{x} \]

Summing forces in the \( y \) direction yields:
\[ N - mg \sin \alpha - k(\sqrt{l^2 + x^2} - l)\frac{\ell}{\sqrt{l^2 + x^2}} = 0 \]

Therefore
\[ \ddot{x} = \frac{1}{m} \left\{ mg \cos \alpha - kx \left(1 - \frac{\ell}{\sqrt{l^2 + x^2}}\right) - \mu_k \frac{\dot{x}}{|\dot{x}|} \left[ mg \sin \alpha + k \ell \left(1 - \frac{\ell}{\sqrt{l^2 + x^2}}\right)\right] \right\} \]
or
\[ \ddot{x} = \frac{1}{m} \left\{ mg \cos \alpha - (kx + k \ell \mu_k \frac{\dot{x}}{|\dot{x}|}) \left(1 - \frac{\ell}{\sqrt{l^2 + x^2}}\right) - \mu_k \frac{\dot{x}}{|\dot{x}|} mg \sin \alpha \right\} \]
2.33 Solution:

\[
\begin{align*}
&k := 40 \quad \mu_k := 0.2 \quad L := 0.2 \quad m := 5 \quad g := 9.81 \quad \alpha := 30 \, \text{deg} \\
i := 0..4000 \\
\Delta t := 0.001 \\
&\begin{pmatrix} v_0 \\ x_0 \end{pmatrix} := \begin{pmatrix} 0.0000 i \\ 0 \end{pmatrix} \\
a(x,v) := \frac{1}{m} \left[ mg \cos(\alpha) - k \left( x + \mu_k L \cdot \frac{v}{|v|} \right) \left( 1 - \frac{L}{\sqrt{L^2 + v^2}} \right) - \mu_k m g \sin(\alpha) \cdot \frac{v}{|v|} \right] \\
&\begin{pmatrix} v_{i+1} \\ x_{i+1} \end{pmatrix} := \begin{pmatrix} v_{i} + a(x_{i}, v_{i}) \Delta t \\ x_{i} + v_{i} \Delta t \end{pmatrix}
\end{align*}
\]
2.34 Solution:

\[ \ddot{x} = +g, \text{ but } v \, dv = +g \, dx, \text{ so that } v_0 = \sqrt{2 \cdot g \cdot \ell} = 62.16 \text{ ft/s} \]

Now the FBD is illustrated in S2.34b and yields \(-kx + mg = m \ddot{x}\), with \(v = v_0\).

Thus \(a = v \frac{dv}{dx} = g - \frac{k}{m} x\)

Integrating both sides yields:

\[ \int_{v_0}^{0} v \, dv = \int_{-d}^{0} \left[ g - \frac{k}{m} x \right] \, dx \]

\[-g \ell = gd - \frac{k}{2m} d^2 \]

\[ d = \frac{-g \pm \sqrt{g^2 + 2kmg}}{k} = 77.893 \text{ ft (positive root)} \]

Therefore \(150 - 60 - 77.9 = 12.1 \text{ ft above the ground.} \)

Also, note from the quadratic formula above (using \(k = c/\ell\)) that \(d\) is proportional to \(\ell\). Thus, \(F_x = kd = \frac{cd}{\ell}\) is independent of \(\ell\). For the numbers given, \(F_x = 5 \cdot 77.9 = 389.5 \text{ lb}\). 

```
2.35 Solution:

\[ cv - mg = 0 \]

```

\[ \text{FIGURE S2.34a,b} \]

From the FBD, the sum of the forces in the \(y\)-direction when the skydiver has reached her terminal velocity (i.e. zero acceleration) yields:

\[ cv - mg = 0 \]
Solving for $c = 4.0 A = \frac{mg}{v}$.

Thus $A = \frac{1}{4.0} \cdot \frac{(60)(0.81)}{9} = 16.4 \text{ m}^2$

2.36 From the FBD of the man there are 3 forces acting on him: gravity, the drag force, which shuts “off” after 10 meters, and the buoyancy force which shuts off when he stops. The force summation yields

$$-F_b - F_c + mg = m\ddot{x}$$

Now $F_c = cv$ from 10 m or $F_c = cv < x - 10 >^0$

Likewise $F_B = -mg$ from 13m or $< x - 13 >^0$

Thus

$$\ddot{x} = -\frac{cv}{m} < x - 10 >^0 - g < x - 13 >^0 + g$$

![FIGURE S2.36]

The force at impact is $m\ddot{x}| = mg - cv$, where $v = \sqrt{2ad}$ and $x = 10$, thus

$F_{\text{impact}} = 70 \cdot 9.81 - 500\sqrt{(2)(10)(9.81)} = -6313 \text{ N}$. The depth beneath the surface that the diver travels is $d = 13.195-10 = 3.195 \text{ m}$ (see Mathcad soln).

\begin{align*}
m &:= 70 & c &:= 500 & \Delta t &:= 0.001 & g &:= 9.81 & \ i &:= 0..5000 \\
a(v,x) &:= g - \frac{c\cdot v}{m} \cdot \Phi(x-10) \cdot \frac{v}{|v|} - g \cdot \Phi(x-13) \\
\begin{pmatrix} v_0 & \vdots & v_i \\ x_0 & \vdots & x_i \end{pmatrix} &:= \begin{pmatrix} 0 & \vdots & 0 \\ 0 & \vdots & 0 \end{pmatrix} \\
\begin{pmatrix} v_{i+1} \\ x_{i+1} \end{pmatrix} &:= \begin{pmatrix} v_i + a(v_i,x_i)\Delta t \\ x_i + v_i\Delta t \end{pmatrix} \\
\end{align*}

\begin{align*}
x_057 &:= 13.194 \\
x_058 &:= 13.195 \\
x_5000 &:= 13.195 \\
\end{align*}
2.37 This can be solved by “hand” or by using the singularity function and numerically integrating.

By hand: for the first 5 seconds:

\[ a(t) = 0.7t, \]
\[ v(t) = \frac{0.7}{2}t^2 \]
\[ s(t) = \frac{0.7}{6}t^3 \]

At \( t = 5 \), we have \( v_5 = 8.75 \text{ m/s}, s_5 = 14.58 \text{ m}. \)

After the first 5 seconds \( a = -2 \text{ m/s} \) and integrating yields (starting the clock over at \( t = 0 \))

\[ v(t) = -2t + v_5 = -2t + 8.75 \]
\[ s(t) = -\frac{2t^2}{2} + 8.75t \]

Solving \( v(t) = 0 \) for \( t \) yields \( t = \frac{8.75}{2} = 4.38 \text{ s} \).

Thus in the second interval \( s(4.38) = 19.14 \text{ m} \) is reached.

The total distance traveled is \( 14.58 + 19.14 = 33.72 \text{ m} \).

This is solved by simple numerical integration in the following Mathcad code.

\[
\begin{align*}
i &:= 0..940 \\
g &:= 9.81 \\
m &:= 1200 \\
\Delta t &:= 0.01 \\
\tau_i &:= i \cdot \Delta t \\
a(t) &:= \Phi(5 - t) \cdot 0.7 \cdot t - \Phi(t - 5) \cdot 2 \\
\begin{bmatrix} v_0 \\
x_0 \end{bmatrix} &:= \begin{bmatrix} 0 \\
0 \end{bmatrix} \\
\begin{bmatrix} v_{i+1} \\
x_{i+1} \end{bmatrix} &:= \begin{bmatrix} v_i + a(t) \cdot \Delta t \\
x_i + v_i \cdot \Delta t \end{bmatrix}
\end{align*}
\]

The total distance traveled is 33.8 meters.
2.38 The free body diagram given in S2.38:

Here $F_d = c\dot{x} < x - 35 >^0 < v >$ is the damping force applied by the mat and $F_k = k(x - 35)^2 < x - 35 >^0$ is the spring force applied by the mat. The equation of motion obtained by summing forces in the vertical direction is just $m\ddot{x} = mg - F_d - F_k$. Two design criteria must be met: total force must be less than 600 lb and the total mat displacement must be less than 6 ft. The following Mathcad code computes the displacement by numerical integration, and can be evaluated for various values of $c$ and $k$ simply by changing the definition statements for these constants. The solution is shown for one possible combination of $c$ and $k$ that satisfy the design criteria.

\[
g := 32.2 \quad m := \frac{120}{g} \quad k := 32 \quad c := 20
\]

\[
a(v, x) := g - \Phi(x - 20) \frac{k}{m} (x - 20)^2 - \Phi(x - 20) \frac{c}{m} v \cdot \Phi(v)
\]

\[
\Delta t := 0.001 \quad i := 0..8000 \quad t_i := i \cdot \Delta t
\]

\[
\begin{pmatrix} v_0 \\ x_0 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} v_{i+1} \\ x_{i+1} \end{pmatrix} := \begin{pmatrix} v_i + a(v_i, x_i) \Delta t \\ x_i + v_i \cdot \Delta t \end{pmatrix}
\]

\[
F_i := m a(v_i, x_i)
\]
2.39 The sum of forces in the vertical direction ignoring gravity is
\[ m\ddot{x} = 50000[< 3 - t >^0 + \sum_{n=1}^{4} < t - 6n >^0 < 6n + 3 - t >] \]
The following Mathcad code integrates the equation of motion and display the
displacement and velocity:

\[
m = 500 \quad F : = 50000 \\
a(t) : = \frac{F}{m} \left[ \Phi(3 - t) + \sum_{n=1}^{4} \left( \Phi(t - 6n) \cdot \Phi(6n + 3 - t) \right) \right] \\
\Delta t : = 0.01 \quad i : = 0 \ldots 2700 \quad t_i : = i \Delta t \\
\begin{pmatrix} v_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix} \begin{pmatrix} v_{i+1} \\ x_{i+1} \end{pmatrix} = \begin{pmatrix} v_i + a(t_i) \Delta t \\ x_i + v_i \cdot \Delta t \end{pmatrix} \\

v_{2700} = 1.604 \times 10^3 \\
x_{2700} = 2.3 \times 10^4 \\

2.40 A FBD of the ball is given in S2.40a.

\[ F \]
\[ m \]
\[ y \]
\[ x \]

**FIGURE S2.40**
The problem is solved by setting up the computational solution and then per-
forming a trial and error procedure to find the required force. The Mathcad
solution is shown for one possible value of \( F \) (roughly the minimum possible
value). The initial conditions are \( v_x(0) = -120 \text{ ft/s} \), \( x(0) = 0 \), \( y(0) = 4 \text{ ft} \) (this
is assumed to be the level of the ball when it is struck, based on an average
batter size and general location of the strike zone) and \( v_y(0) = 0 \). The force
required is found to be about 128 lb.
A free body diagram of the box is given in the figure.

The forces are \( F_k = kx < x - 4 >^0 \) \( f = \mu_k \cos \theta \frac{\omega}{|v|} \)

The FBD yields the following two equations

\[ \sum F_y : \quad N - mg \cos 45^\circ = 0 \]
\[ \sum F_x : \quad m\ddot{x} = -f - F_k + mg \sin 45^\circ \]
The following Mathcad code integrates to find the solution.

\[ i := 0 \ldots 5000 \quad \Delta t := 0.001 \quad t_i := i \cdot \Delta t \]

\[ \mu := 0.6 \quad \theta := 45 \text{ deg} \quad m := 2 \quad g := 9.81 \quad k := 500 \]

\[ a(v, x) := \frac{1}{m} \left[ m \cdot g \cdot \left( \sin(\theta) - \mu \cdot \cos(\theta) \cdot \frac{v}{|v|} \right) - \Phi(x - 4) \cdot [k \cdot (x - 4)] \right] \]

\[
\begin{bmatrix}
  v_0 \\
  x_0
\end{bmatrix} :=
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  v_{i+1} \\
  x_{i+1}
\end{bmatrix} :=
\begin{bmatrix}
  v_i + a(v_i, x_i) \cdot \Delta t \\
  x_i + v_i \cdot \Delta t
\end{bmatrix}
\]

![Graph showing the solution](image-url)
function yprime = ds2pt41(t,y)  
% friction force
if y(2) == 0,  
    Ef = 0;  
else  
    Ef = 4.162*y(2,:)/abs(y(2,:));  
end  
% spring force
if y(1) <= 0,  
    Fk = 0;  
else  
    Fk = 250*y(1,:);  
end  
yprime(2,:) = -Ef-Fk+6.947;  
return

2.42 Working with the free body diagram of the previous problem replace the forces $F_k$ with $F_k + F_c$ where $F_c = cv(x - 4)$. Thus the equation of motion can be written as: 
\[ ma = mg\sin(\theta - \mu_k \cos \theta \frac{v}{|v|}) - < x - 4 >^\circ (k(x - 4) + cv) \]

The following MATLAB code can be used to solve for the velocity and displacement using ODE for plotting the response.

function yprime = ds2pt41(t,y)  
% friction force
if y(2) == 0,  
    Ff = 0;  
else  
    Ef = 4.162*y(2,:)/abs(y(2,:));  
end  
% spring and damper force
if y(1) <= 0,  
    Fc = 0;  
else  
    Fc = 12*y(2,:);  
end  
yprime(2,:) = -Ef-Fc-Fk+6.947;  
return
Next the Mathcad solution is presented, complete with the plots:

\[
\begin{align*}
k &:= 500 & \mu k &:= 0.6 & m &:= 2 & g &:= 9.81 & \theta &:= 45\text{ deg} \\
i &:= 0..5000 & \Delta t &:= 0.001 & t_i &:= i \Delta t \\
 a(v, x, c) &:= \frac{1}{m} \left[m g \left(\sin(\theta) - \mu k \cos(\theta) \frac{v}{|v|}\right)\right] - \Phi(x - 4) \cdot [k(x - 4) + c \cdot v] \\
 \begin{bmatrix} v_0 \\ x_0 \end{bmatrix} &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} v_{i+1} \\ x_{i+1} \end{bmatrix} &:= \begin{bmatrix} v_i + a(v_i, x_i, 20) \Delta t \\ x_i + v_i \Delta t \end{bmatrix}
\end{align*}
\]

These codes are run several times for different values of \( c \) until one is found that keeps the box from losing contact with the spring. A value of \( c = 20 \) very nearly keeps it in contact, but a value as high as \( c = 200 \) is needed to truly keep the box in contact with the spring.
2.43 Assuming positive in the direction of the expanding bag:

\[ F_d = 5 \text{ Ns/m} \cdot v_{H/B} \text{ m/s} \]
\[ v_{H/B} = 30 + v_B \]
\[ 5(30 + v_B) = 500 \]
\[ v_B = 70 \text{ m/s} \]

2.44 Solution:

From the free body diagram, Newton’s law in the vertical direction yields \( F = mg - 10H(x - 5)(x - 5) - 0.5|v|v \) Where the Heaviside function indicates that the spring forces does not act until \( x = 5 \).

The MATLAB code for solving this consists of the following m-file, then using ODE and PLOT to compute and plot the solution \( x(t) \). The m-file is

function yprime = ds2pt41(t,y)
    yprime(1,:) = y(2,:);
    \% viscous force
    Fd = 0.1*y(2,:)*abs(y(2,:));
    \% spring force
    if y(1) < =0,
        Fk = 0;
    else
        Fk = 2*y(1,:);
    end
    yprime (2,:) = -Fd-Fk+9.81;
    return
The Mathcad equivalent is given next along with the plot of $x(s)$ vs. $t$:

$$m := 5 \quad k := 10 \quad c := 0.5 \quad g := 9.81$$

$i := 0..10000$

$$\Delta t := 0.001$$

$t_i := i \cdot \Delta t$

$$a(v, x) := m \cdot g - k \cdot \Phi(x - 5) \cdot (x - 5) - c \cdot v \cdot |v|$$

$$
\begin{bmatrix}
\dot{v}_0 \\
\dot{x}_0
\end{bmatrix} :=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
\dot{v}_{i+1} \\
\dot{x}_{i+1}
\end{bmatrix} :=
\begin{bmatrix}
v_i + a(v_i, x_i) \cdot \Delta t \\
x_i + v_i \cdot \Delta t
\end{bmatrix}
$$

---

2.45 Solution:

$$\rho = 100 \text{ ft} \quad v = 60 \text{ mph} = 88 \text{ ft/s}$$

$$\sum F_n = ma_n = \frac{mv^2}{\rho}$$

$$N - 130 = \frac{130 \cdot v^2}{32.2 \cdot 100}$$
Therefore \( N - 130 = \left( \frac{130}{32.2} \right) \frac{88^2}{100} \)

\( N = 443 \text{ lb up} \)

2.46 Solution:

![Figure S2.46](image)

To fly the hill \( N = 0 \)

\[ mg = m \frac{v^2}{500} \]

\[ v = \sqrt{32.2 \times 500} = 126.9 \text{ ft/s} = 86.5 \text{ mph} \]

2.47 Solution:

![Figure S2.47](image)

\[ N \cos \beta - mg - f \sin \beta = 0, \quad N \sin \beta + f \cos \beta = \frac{mv^2}{\rho} \]

\[ f = \mu_s N \quad N = \frac{mg}{\cos \beta - \mu \sin \beta} \]

\[ \frac{mv^2}{\rho} = mg \frac{\sin \beta + \mu \cos \beta}{\cos \beta - \mu \sin \beta} \]

\[ v = \sqrt{\rho g \frac{\sin \beta + \mu \cos \beta}{\cos \beta - \mu \sin \beta}} \]

2.48 Solution: 25 mph = 36.7 ft/s. Therefore \( \frac{36.7^2}{100} = 32.2 \left[ \frac{\sin \beta + 0.3 \cos \beta}{\cos \beta - 0.3 \sin \beta} \right] \)

\[ 0.418(\cos \beta - 0.3 \sin \beta) = \sin \beta + 0.3 \cos \beta \]

\[ \beta = 6.02^\circ \]
2.49 Solution:

\[ a_{\text{max}} = 5g = \frac{v^2}{p} \quad v = 700 \text{ mph} = 1027 \text{ ft/s} \]
\[ \rho = \frac{(1027)^2}{5(32.2)} = 6551 \text{ ft} \]

2.50 Solution:

\[ mg \sin \beta = mR \ddot{\beta} \quad wdw = \frac{4}{R} \sin \beta d\beta \]
\[ \frac{w^2}{a} \bigg|_{w_0} = \frac{g}{R} (-\cos \theta)_0 \]
\[ w^2 = \frac{2g}{R} (1 - \cos \beta) + w_0^2 \]

At separation:
\[ mg \cos \beta = m \frac{v^2}{R} \quad \frac{v^2}{R^2} = \frac{2g}{R} (1 - \cos \beta) + \frac{v_0^2}{R^2} \]
\[ g \cos \beta = 2g(1 - \cos \beta) + \frac{v_0^2}{R} \quad 3g \cos \beta = 2g + \frac{v_0^2}{R} \]
\[ \beta = \cos^{-1} \left[ \frac{2}{3} + \frac{v_0^2}{3gR} \right] \]
2.51 Solution:

The following equations can be written:

\[ m g \cos \theta = m R \ddot{\theta} \quad (1) \]
\[ N - m g \sin \theta = m R \dot{\theta}^2 \quad (2) \]

Eq (1) can be rewritten as \( \omega \frac{d\omega}{d\theta} = \frac{g}{R} \cos \theta \) (where \( \omega = \dot{\theta} \)).

Integrating this gives:

\[ \frac{\omega^2}{2} = \frac{g}{R} (\sin \theta - \sin \theta_0) \]

Thus:

\[ \omega = \sqrt{\frac{2g}{R} (\sin \theta - \sin \theta_0)} \]

and

\[ v = \sqrt{2gR (\sin \theta - \sin \theta_0)} \]

2.52 Solution:

\[ m R \ddot{\theta} = m g \cos \theta - f \frac{\dot{\theta}}{|\dot{\theta}|} \]
\[ -m g \sin \theta + N = m R \dot{\theta}^2 \]

Therefore \( m R \ddot{\theta} = m g \cos \theta - \mu_k \frac{\dot{\theta}}{|\dot{\theta}|} \left[ m g \sin \theta + m R \dot{\theta}^2 \right] \)

Canceling the mass

\[ \ddot{\theta} = \frac{g}{R} \cos \theta - \frac{\mu_k}{R |\dot{\theta}|} \left[ g \sin \theta + R \dot{\theta}^2 \right] \]
2.53 Solution:

The mass cancels out of the equation of motion and values have been assumed for the radius and the coefficient of kinetic friction. The Mathcad code is:

\[
i := 0 \ldots 2000 \\
\Delta t := 0.001 \\
g := 9.81 \\
\mu_k := 0.15 \\
R := 0.3 \\
t_i := i \cdot \Delta t
\]

\[
\alpha(\omega, \theta) := \frac{g}{R} \cos(\theta) - \frac{\mu_k \omega}{|\omega|} \left( g \cdot \sin(\theta) + R \cdot \omega^2 \right)
\]

\[
\begin{bmatrix}
\omega_0 \\
\theta_0
\end{bmatrix} :=
\begin{bmatrix}
0 \\
10 \cdot \text{deg}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\omega_{i+1} \\
\theta_{i+1}
\end{bmatrix} :=
\begin{bmatrix}
\omega_i + \alpha(\omega_i, \theta_i) \cdot \Delta t \\
\theta_i + \omega_i \cdot \Delta t
\end{bmatrix}
\]

The equivalent code in MATLAB is:

```matlab
function yprime = ds2pt53(t,y)
g = 9.81; R = 1; gg = g/R; mu = 0.07;
yprime(1) = y(2);
if y(2)==0
    yprime(2) = -mu*(y(2)^2+gg*sin(y(1)))+gg*cos(y(1));
else
    yprime(2) = -mu*(y(2)^2+gg*sin(y(1)))*y(2)/abs(y(2))+gg*cos(y(1));
end
return
```

The equivalent code in MATLAB is:

```matlab
function yprime = ds2pt53(t,y)
g = 9.81; R = 1; gg = g/R; mu = 0.07;
yprime(1) = y(2);
if y(2)==0
    yprime(2) = -mu*(y(2)^2+gg*sin(y(1)))+gg*cos(y(1));
else
    yprime(2) = -mu*(y(2)^2+gg*sin(y(1)))*y(2)/abs(y(2))+gg*cos(y(1));
end
return
```
2.54 See the free body diagram given in Sample Problem 2.12. It is given that:

\[ F_s = kR (2 \cos \frac{\theta}{2} - 1) \]
\[ F_f = \mu_k |N| \frac{\dot{\theta}}{|\dot{\theta}|} \]

From Sample 2.12, the FBD yields upon summing forces in the normal and tangential directions

\[ \sum F_n = -N + mg \sin \theta + F_s \cos \frac{\theta}{2} = mR\ddot{\theta}^2 \]
\[ N = mg \sin \theta + kR (2 \cos \frac{\theta}{2} - 1) \cos \frac{\theta}{2} - mR\ddot{\theta}^2 \]
\[ \sum F_t = -F_f + F_s \sin \frac{\theta}{2} - mg \cos \theta = mR\ddot{\theta} \]

The equation of motion is thus:

\[ mR\ddot{\theta} = -\mu_k \left| mg \sin \theta + kR (2 \cos \frac{\theta}{2} - 1) \cos \frac{\theta}{2} - mR\ddot{\theta}^2 \right| \frac{\dot{\theta}}{|\dot{\theta}|} + kR (2 \cos \frac{\theta}{2} - 1) \sin \frac{\theta}{2} - mg \cos \theta \]
\[ \ddot{\theta} = -\mu_k \left| \frac{\dot{\theta}}{|\dot{\theta}|} \left( \frac{\theta}{R} \sin \theta + \frac{kR}{m} (2 \cos \frac{\theta}{2} - 1) \cos \frac{\theta}{2} - \ddot{\theta}^2 \right) \right| + \frac{kR}{m} (2 \cos \frac{\theta}{2} - 1) \sin \frac{\theta}{2} - \frac{kR}{m} \cos \theta \]

When the slider is in equilibrium, \( \ddot{\theta} = 0 \) and \( \dot{\theta} = 0 \). The Mathcad solution follows.

The solution shown here is for a coefficient of kinetic friction of 0.18, a value that was chosen for a typical response. The spring would be unloaded when \( \theta = 120^\circ \) but there would still be the gravitational acceleration acting on the slider. It may be observed that the slider is oscillating about a slighter higher angle (roughly 123\(^\circ\)), which is the equilibrium angle.

Note the slow start of the slider as it overcomes the friction force. The spring force is high when \( \theta = 30^\circ \) but this contributes to a high normal force and therefore a high friction force. If the coefficient of kinetic friction is too high, could the problem be such that friction drives the motion? No, this is an impossibility. In this formulation the friction term in the equation of motion is multiplied by \( \dot{\theta}/|\dot{\theta}| \) to make sure the friction always opposes motion; this term is zero when the angular velocity is zero so friction will never initiate motion. To determine whether motion would occur in a particular case, determine whether the spring force minus the gravitational force in the initial configuration is greater than the coefficient of static friction times the normal force. That would be a separate calculation as the friction force in this equation of motion is zero until motion starts.
\[\begin{align*}
m &= 1 \\
k &= 600 \\
R &= 0.2 \\
g &= 9.81 \\
\mu_k &= 0.18 \\
N(\omega, \theta) &= m \cdot g \cdot \sin(\theta) + k \cdot R \left(2 \cdot \cos\left(\frac{\theta}{2}\right) - 1\right) \cdot \cos\left(\frac{\theta}{2}\right) - m \cdot R \cdot \omega^2 \\
\alpha(\omega, \theta) &= \frac{-g}{R} \cdot \cos(\theta) + \frac{k}{m} \left(2 \cdot \cos\left(\frac{\theta}{2}\right) - 1\right) \cdot \sin\left(\frac{\theta}{2}\right) - \frac{\mu_k}{m \cdot R} \cdot N(\omega, \theta) \left|\omega\right| \\
i &= 0..15000 \\
\Delta t &= 0.0001 \\
t_i &= i \cdot \Delta t \\
\left(\omega_0 \at \theta_0\right) &= \left(0 \at \frac{\pi}{6}\right) \\
\left(\omega_{i+1} \at \theta_{i+1}\right) &= \left(\omega_i + \alpha(\omega_i, \theta_i) \Delta t \at \theta_i + \omega_i \Delta t\right) \\
\end{align*}\]

\[\begin{align*}
f(\theta) &= \frac{-g}{R} \cdot \cos(\theta) + \frac{k}{m} \left(2 \cdot \cos\left(\frac{\theta}{2}\right) - 1\right) \cdot \sin\left(\frac{\theta}{2}\right) \\
x &= 120 \cdot \text{deg} \\
\text{root}(f(x), x) \cdot \text{deg} &= 123.35
\end{align*}\]
2.55 We are given $\alpha = \text{constant}$ and so $\omega = \alpha t$.

\[ f = m(r\alpha \hat{e}_z + r\omega^2 \hat{e}_n) \] is the total force. The total acceleration is

\[
\begin{align*}
\mu_s mg &= m\sqrt{(r\alpha)^2 + (r\omega^2)^2} \\
\mu_s &= \frac{1}{g} \sqrt{(r\alpha)^2 + (r(\alpha t)^2)} = \frac{r\alpha}{g} \sqrt{1 + \alpha^2 t^4}
\end{align*}
\]

2.56 Solution:

\[ R_A + R_B = \ell \\
T = m_B R_B \omega^2, \ T = m_A R_A \omega^2 \]
Therefore $m_B R_B = m_A R_A$.

\[
\begin{align*}
R_A &= \ell - R_B \\
R_B &= \frac{m_B}{m_B} \ell - \frac{m_B}{m_B} R_B \\
R_B &= \frac{m_A}{m_A + m_B} \ell \\
R_A &= \frac{m_B}{m_A + m_B} \ell
\end{align*}
\]
2.57 Solution:

\[
\begin{align*}
\text{Sum forces along } \hat{t}: \quad & N \cos \theta - mg = 0 \\
\text{Sum forces along } \hat{n}: \quad & N \sin \theta = m R \omega^2 \sin \theta \\
\text{Therefore } & N = m R \omega^2, \quad m R \omega^2 \cos \theta = mg, \quad \theta = \cos^{-1} \left( \frac{g}{R \omega^2} \right).
\end{align*}
\]

2.58 First-determine the required velocity at top.

\[
\begin{align*}
N + mg &= m \frac{v^2}{R}, \quad N = \frac{1}{2} mg \\
v &= \sqrt{\frac{3}{2} g R} \quad \text{or} \quad \omega_T = \sqrt{\frac{3 g}{2 R}}.
\end{align*}
\]
From S2.58b:

\[ mR \ddot{\theta} = -mg \sin \theta \quad \text{(sum of tangential forces)} \]

\[ \int_{\omega_0}^{\omega} \omega d\omega = \int_0^\pi \frac{a}{R} \sin \theta d\theta \]

\[ \frac{\omega^2 - \omega_0^2}{2} = + \frac{a}{R} \cos \theta \Big|_0^\pi = -\frac{2a}{R} \]

\[ \omega_0^2 = \frac{4a}{R} + \frac{3}{2} \frac{a}{R} = \frac{11a}{2R} \]

\[ v_0 = \sqrt{\frac{11}{2} gR} \]

2.59 From Problem 2.58

\[ N - mg = \frac{mg_0^2}{R} = m \cdot \frac{11}{2} g \quad \text{or} \quad N = mg + \frac{11}{2} mg \]

\[ N = \frac{13}{2} mg \]

6.5 times body weight could realistically cause spinal injuries.
2.60 See free body diagram shown in the solution to problem 2.61 (S2.61).

From the solution to problem 2.61:
\[ \ddot{y}_A = \frac{\ell \dot{\theta}^2 \sin \theta + g \cos \theta \sin \theta}{\gamma + \sin^2 \theta}, \]
where \[ \gamma = \frac{m_A}{m_B} \]
For \( \ell = 0.2 \) m, \( \theta = 30^\circ \), \( \dot{\theta}(0) = 0 \), \( m_A = 3 \) kg and \( m_B = 2 \) kg:
\[ \ddot{y}_A = 2.427 \text{ m/s} \] (acceleration of block A).

The acceleration of block B is:
\[ \mathbf{a}_B = a_A \hat{j} + \ell \ddot{\theta} \cos \theta \hat{j} - \ell \ddot{\theta} \sin \theta \hat{i} \]
From the solution to problem 2.61:
\[ (m_A + m_B \sin^2 \theta) \ddot{\theta} + m_B \dot{\theta}^2 \cos \theta \sin \theta + (m_A + m_B) \frac{g}{\ell} \sin \theta = 0 \]
For the numbers given:
\[ \ddot{\theta} = -35.036 \text{ rad/s}^2. \]
And:
\[ \mathbf{a}_B = 2.427\hat{j} - 6.068\hat{j} + 3.504\hat{i} = 3.504\hat{i} - 3.641\hat{j} \text{ m/s}. \]
2.61 The FBD’s are given in the figure:

In the y-direction for block A:

(1)  \( T \sin \theta = m_A \ddot{y}_A \)

and in the radial and tangential directions for block B:

(2)  \( -T + m_B g \cos \theta = m_B(-\ell \dot{\theta}^2 + \ddot{y}_A \sin \theta) \)

(3)  \(-m_B g \sin \theta = m_B \ell \dot{\theta} + m_B \ddot{y}_A \cos \theta \)

Substitution of (2) into (1) yields

\[
\sin \theta \{ m_B \ell \dot{\theta}^2 - m_B \ddot{y}_A \sin \theta + m_B g \cos \theta \} = m_A \ddot{y}_A
\]

or

\[
\ell \dot{\theta}^2 \sin \theta + g \cos \theta \sin \theta = (\frac{m_A}{m_B} + \sin^2 \theta) \ddot{y}_A
\]

Thus we have

(4)  \( \ddot{y}_A = \frac{\ell \dot{\theta}^2 \sin \theta + g \cos \theta \sin \theta}{\gamma + \sin^2 \theta} \)

where

\( \gamma = \frac{m_A}{m_B} \)

Substitute 4 into 3:

\[
\ell \ddot{\theta} + \frac{\ell \dot{\theta}^2 \sin \theta + g \cos \theta \sin \theta}{\gamma + \sin^2 \theta} \cos \theta = -g \sin \theta.
\]

\[
(\gamma + \sin^2 \theta) \ddot{\theta} + \dot{\theta}^2 \cos \theta \sin \theta + \frac{\dot{\theta}^2}{\gamma} \cos^2 \theta \sin \theta + \frac{\dot{\theta}^2}{\gamma} \sin \theta + \frac{\dot{\theta}^2}{\gamma} \sin^3 \theta = 0
\]

which can also be written as:

\[
(1 + \frac{\sin^2 \theta}{\gamma}) \ddot{\theta} + \frac{1}{\gamma} \dot{\theta}^2 \cos \theta \sin \theta + \frac{\dot{\theta}^2}{\gamma} \cos^2 \theta \sin \theta + \frac{\dot{\theta}^2}{\gamma} \sin \theta + \frac{\dot{\theta}^2}{\gamma} \sin^3 \theta = 0
\]

When \( m_A >> m_B \) (\( \gamma \) very large) this becomes the pendulum equation:

\( \ddot{\theta} + \frac{\dot{\theta}^2}{\gamma} \sin \theta = 0 \)

Resubstituting for \( \gamma \) into the original equation and simplifying:

\[
(m_A + m_B \sin^2 \theta) \ddot{\theta} + m_B \dot{\theta}^2 \cos \theta \sin \theta + (m_A + m_B) \frac{\dot{\theta}^2}{\ell} \sin \theta = 0
\]
2.62 The free body diagrams are given in S2.62:

![Free Body Diagram](image)

**FIGURE S2.62**

From the solution to problem 2.63 for $\ddot{\theta}$, with $\theta = 0$ and $\dot{\theta} = 0$:

$$\ddot{\theta} = \frac{-g \sin \beta \cos \beta (m_A + m_B)}{\ell (m_A + m_B \sin^2 \beta)}$$

Writing the equation of motion in the $x$-direction for block A in the coordinate system shown in S2.62:

$$m_A a_A = m_A g \sin \beta + T \sin \beta$$

Writing the equation of motion in the $y$-direction for block B in the coordinate system shown in S2.62:

$$T \cos \beta - m_B g \cos \beta = m_B \ell \ddot{\theta} \sin \beta$$

Solving the previous equation for $T$:

$$T = m_B g + m_B \ell \ddot{\theta} \tan \beta$$

Substituting this gives the acceleration of block A (which is in the $x$-direction):

$$a_A = \frac{m_A + m_B}{m_A} g \sin \beta + \frac{m_B}{m_A} \ell \ddot{\theta} \sin \beta \tan \beta$$

And the acceleration of B is:

$$a_B = a_A \hat{i} + \ell \ddot{\theta} \cos \beta \hat{i} + \ell \ddot{\theta} \sin \beta \hat{j}$$
2.63 Solution:

\[ T \sin(\beta + \theta) + m_A g \sin \beta = m_A a_A \]
\[ -T \sin \theta = m_B a_{Bx} \]
\[ T \cos \theta - m_B g = M_B a_{By} \]
\[ a_B = a_A + a_{B/A} \]
\[ a_A = a_A(\cos \beta \hat{i} - \sin \beta \hat{j}) + \ell \dot{\theta}^2 (-\sin \theta \hat{i} + \cos \theta \hat{j}) \]

1) \[ T \sin(\beta + \theta) + m_A g \sin \beta = m_A a_A \]

2) \[ -T \sin \theta = m_B [a_A \cos \beta + \ell \dot{\theta} \cos \theta - \ell \dot{\theta}^2 \sin \theta] \]
\[ T \left[ \sin \theta + \frac{m_B}{m_A} \sin(\beta + \theta) \cos \beta \right] = -m_B g \sin \beta - m_B \ell \dot{\theta} \cos \theta + m_B \ell \dot{\theta}^2 \sin \theta \]
\[ T = \frac{-m_B [g \sin \beta \cos \beta + \ell \dot{\theta} \cos \theta - \ell \dot{\theta}^2 \sin \theta]}{\sin \theta + \frac{m_B}{m_A} \sin(\beta + \theta) \cos \beta} \]

3) \[ T \cos \theta = m_B g - m_B [a_A \sin \beta - \ell \dot{\theta} \sin \theta - \ell \dot{\theta} \cos \theta] \]
\[ T \left[ \cos \theta + \frac{m_B}{m_A} \sin(\beta + \theta) \sin \beta \right] = m_B [g(1 - \sin^2 \beta) + \ell \dot{\theta} \sin \theta + \ell \dot{\theta}^2 \cos \theta] \]
\[ T = \frac{m_B [g(1 - \sin^2 \beta) + \ell \dot{\theta} \sin \theta + \ell \dot{\theta}^2 \cos \theta]}{\cos \theta + \frac{m_B}{m_A} \sin(\beta + \theta) \sin \beta} \]

Equating two expressions for \( T \)
\[ -[\cos \theta + \frac{m_B}{m_A} \sin(\beta + \theta) \sin \beta] \cdot [g \sin \beta \cos \beta + \ell \dot{\theta} \cos \theta - \ell \dot{\theta}^2 \sin \theta] = [\sin \theta + \frac{m_B}{m_A} \sin(\beta + \theta) \cos \beta] \cdot [g \cos^2 \beta + \ell \dot{\theta} \sin \theta + \ell \dot{\theta}^2 \cos \theta] \]

Therefore \[ g \sin \beta \cos \beta \cos \theta + \ell \dot{\theta} \cos^2 \theta - \ell \dot{\theta}^2 \sin \theta \cos \theta \]
\[ + g \cos^2 \beta \sin \theta + \ell \dot{\theta} \sin^2 \theta + \ell \dot{\theta} \sin \theta \cos \theta \]
\[ + \frac{m_B}{m_A} \sin(\beta + \theta) [g \sin^2 \beta \cos \beta + \ell \dot{\theta} \cos \theta \sin \beta - \ell \dot{\theta}^2 \sin \theta \sin \beta] \]
\[ + \frac{m_B}{m_A} \sin(\beta + \theta) [g \cos^2 \beta \cos \beta + \ell \dot{\theta} \sin \theta \cos \beta + \ell \dot{\theta}^2 \cos \theta \cos \beta] \]
\[ g \sin(\beta + \theta) \cos \beta + \ell \ddot{\theta} + \frac{m_B}{m_A} \sin(\beta + \theta) [g \cos \beta + \ell \dot{\theta} \sin(\beta + \theta) + \ell \dot{\theta}^2 \cos(\beta + \theta)] = 0 \]
\[ g \sin(\beta + \theta) \cos(\frac{m_A + m_B}{m_A}) + \ell \dot{\theta} \left[ 1 + \frac{m_B}{m_A} \sin^2(\beta + \theta) \right] \]
\[ + \ell \dot{\theta}^2 \frac{m_B}{m_A} \sin(\beta + \theta) \cos(\beta + \theta) = 0 \]
\[ \dot{\theta} = \frac{-g \sin(\beta + \theta) \cos(\frac{m_A + m_B}{m_A}) + \ell \dot{\theta}^2 m_B \sin(\beta + \theta) \cos(\beta + \theta)}{\ell [m_A + m_B \sin^2(\beta + \theta)]} \]

2.64 Solution: (in Mathcad)

\[
\begin{align*}
i &:= 0 \ldots 2000 \\
\Delta t &:= 0.001 \\
t_i &:= i \cdot \Delta t \\
m_A &:= 3 \\
m_B &:= 2 \\
g &:= 9.81 \\
\beta &:= 0 \\
L &:= 0.5 \\
dd\theta(\theta, \theta) &:= \left[ (m_A + m_B) \cdot g \cdot \sin(\beta + \theta) \cdot \cos(\beta) + m_B \cdot L \cdot \dot{\theta}^2 \cdot \sin(\beta + \theta) \cdot \cos(\beta + \theta) \right] \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{align*}
\]

\[
\begin{align*}
\left[ \frac{d\theta_0}{\theta_0} \right] &:= \left[ \begin{array}{c} 0 \\
20\text{-deg} \end{array} \right] \\
\left[ \frac{d\theta_i + 1}{\theta_i + 1} \right] &:= \left[ \begin{array}{c} d\theta_i + dd\theta(\theta_i, \theta_i) \cdot \Delta t \\
\theta_i + d\theta_i \cdot \Delta t \end{array} \right]
\end{align*}
\]

Angle-time relationship

\[
\begin{align*}
\text{deg} &\quad \text{deg} \\
\theta_i &\quad \Delta t \\
0 &\quad 0.5 \\
0 &\quad 1 \\
0 &\quad 1.5 \\
0 &\quad 2
\end{align*}
\]

\[t_i \text{ time s} \]
2.65 Solution: (in Mathcad)

\[ i := 0..2000 \]
\[ \Delta t := 0.001 \]
\[ t_i := i \cdot \Delta t \]
\[ m_A := 3 \]
\[ m_B := 2 \]
\[ g := 9.81 \]
\[ \beta := 30 \text{ deg} \]
\[ L := 0.5 \]

\[ d\theta(t, \theta) := \frac{-\left( m_A + m_B \right) \cdot g \sin(\beta + \theta) \cdot \cos(\beta) + m_B \cdot L \cdot d\theta^2 \cdot \sin(\beta + \theta) \cdot \cos(\beta + \theta)}{L \cdot \left( m_A + m_B \cdot \sin(\beta + \theta)^2 \right)} \]

\[ \begin{bmatrix} d\theta_0 \\ \theta_0 \end{bmatrix} := \begin{bmatrix} \theta_0 \\ 20 \text{-deg} \end{bmatrix} \]

\[ \begin{bmatrix} \theta_{i+1} \\ \theta_{i+1} \end{bmatrix} := \begin{bmatrix} \theta_i + d\theta \cdot \Delta t \\ \theta_{i+1} + d\theta \cdot \Delta t \end{bmatrix} \]

Angle-time relationship

2.66 Solution:

\[ a_n = 3 \cdot g = \frac{v^2}{p} \quad v = 200 \text{ mph} = 293 \text{ ft/s} \]
\[ p = \frac{(293)^2}{96.6} = 889 \text{ ft} \]

From the solution of 2.47:

\[ a_n = \frac{g \left( \sin \beta + \mu \cos \beta \right)}{\cos \beta - \mu \sin \beta} \cdot 3 \left( \cos \beta - \mu \sin \beta \right) - \left( \sin \beta + \mu \cos \beta \right) = 0 \]
\[-(3\mu + 1) \sin \beta + (3 - \mu) \cos \beta = 0 \]

Therefore \( \tan \beta = \frac{3-\mu}{3\mu+1} \)

\[ \mu = 0.85 \]
\[ \beta = 31.2^\circ \]
The Mathcad code to solve this problem is:

\[ N := 917 \quad n := 0 \ldots N \quad \Delta t := 0.001 \quad t_n := n \cdot \Delta t \quad L := 3 \quad g := 9.81 \]

\[
\begin{pmatrix}
 v_0 \\
 s_0 \\
 \theta_0 \\
 d\theta_0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 0 \\
 -70\text{-deg} \\
 0
\end{pmatrix}
\begin{pmatrix}
 v_{n+1} \\
 s_{n+1} \\
 \theta_{n+1} \\
 d\theta_{n+1}
\end{pmatrix}
\]

\[
v_n - \left( g \cdot \sin(\theta_n) + 0.3 \left[ g \cdot \cos(\theta_n) + d\theta_n \cdot (v_n)^2 \right] \right) \Delta t
\]

\[
s_n + v_n \cdot \Delta t
\]

\[-70\text{-deg} \cdot \left[ 1 - \left( \frac{s_n}{L} \right)^3 \right] + 3.70\text{-deg} \cdot \frac{(s_n)^3}{L^3}
\]

\[ s_N = 3.001 \quad v_N = 4.227 \quad t_N = 0.917 \]

\[ N := 817 \quad n := 0 \ldots N \quad \Delta t := 0.001 \quad t_n := n \cdot \Delta t \quad L := 3 \quad g := 9.81 \]

\[
\begin{pmatrix}
 v_0 \\
 s_0 \\
 \theta_0 \\
 d\theta_0
\end{pmatrix}
= 
\begin{pmatrix}
 0 \\
 0 \\
 -70\text{-deg} \\
 0
\end{pmatrix}
\begin{pmatrix}
 v_{n+1} \\
 s_{n+1} \\
 \theta_{n+1} \\
 d\theta_{n+1}
\end{pmatrix}
\]

\[
v_n - \left( g \cdot \sin(\theta_n) + 0.0 \left[ g \cdot \cos(\theta_n) + d\theta_n \cdot (v_n)^2 \right] \right) \Delta t
\]

\[
s_n + v_n \cdot \Delta t
\]

\[-70\text{-deg} \cdot \left[ 1 - \left( \frac{s_n}{L} \right)^3 \right] + 3.70\text{-deg} \cdot \frac{(s_n)^3}{L^3}
\]

\[ s_N = 2.999 \quad v_N = 6.668 \quad t_N = 0.817 \]

The child’s velocity is 4.227 m/s at the bottom of the slide when the coefficient of kinetic friction is 0.3, and 6.668 m/s when there is no friction. It takes the child 0.917 seconds to reach the bottom of the slide when the coefficient of kinetic friction is 0.3, and 0.817 seconds when there is no friction.
The MATLAB code to solve the problem is as follows:

```matlab
function yprime = ds2pt67(t,y)
    s = y(1); % position
    s_ = y(2); % speed
    L = 3; % length of slide
    mu = 0.3; % kinetic coefficient of friction
    g = 9.81; % acceleration due to gravity
    theta = -1.222*(1-(s/L)^3); % angle
    theta_ = 1.222*3*(s/L)^2/L; % dtheta/ds
    % differential equations
    yprime(1,:) = s_;
    yprime(2,:) = -mu*(s_^2*theta_ + g*cos(theta))*sign(s_) - g*sin(theta);
    % v_x and v_y
    yprime(3,:) = s_*cos(theta);
    yprime(4,:) = s_*sin(theta);
    return
```

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2.68 The Mathcad code to solve the problem:

The height of the slide must first be determined numerically as shown below. The height was originally chosen to be 3 m, and was reduced to 2.854 m so that the bottom would be 0.6 m from the ground.

\[
x_0 := 0 \\
y_0 := 2.854 \\
s := 0, 0.1 \ldots 3 \\
\theta(s) := -70\text{-deg} \left(1 - \left(\frac{s}{3}\right)^3\right) \\
x(s) := x_0 + \int_0^s \cos(\theta(\zeta)) \, d\zeta \\
y(s) := y_0 + \int_0^s \sin(\theta(\zeta)) \, d\zeta \\
y(3) = 0.6
\]

The solution for the child falling from the top is straightforward as the only force acting on the child is gravitational acceleration.

\[
\ddot{y}(t) = -g \, \text{m/s}^2 \\
\dot{y}(t) = -gt \, \text{m/s} \\
y(t) = -\frac{gt^2}{2} + 2.854 \, \text{m}
\]

The time when the child hits the ground is \( t = 0.763 \) s and the velocity is 7.485 m/s.
2.69 Solution:

\[ \begin{align*}
N & := 6263 \\
g & := 9.81 \\
\mu & := 0.5 \\
\alpha & := 47 \cdot \text{deg} \\
L & := 46 \\
i & := 0 \ldots N \\
\Delta t & := 0.001 \\
t_i & := i \cdot \Delta t
\end{align*} \]

\[ a(\theta, d\theta, v) := -g \cdot \sin(\theta) - \mu \cdot (g \cdot \cos(\theta) + v^2 \cdot d\theta) \]

\[ \begin{bmatrix} v_{i+1} \\ s_{i+1} \\ \theta_{i+1} \\ d\theta_{i+1} \end{bmatrix} := \begin{bmatrix} v_i + a(\theta_i, d\theta_i, v_i) \cdot \Delta t \\ s_i + v_i \cdot \Delta t \\ \theta_i + a(\theta_i, d\theta_i, v_i) \cdot \Delta t \\ d\theta_i + v_i \cdot \Delta t \end{bmatrix} - \alpha \cdot \begin{bmatrix} 1 - \frac{(s_i)^2}{L} \\ 2 \cdot \alpha \cdot \frac{s_i}{L^2} \end{bmatrix} \]

\[ t_{6263} = 6.263 \]

\[ v_{6263} = 3.453 \]

\[ s_{6263} = 46.005 \]

\[ \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} := \begin{bmatrix} 0 \\ 20 \end{bmatrix} \]

\[ \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + \cos(\theta_i) \cdot (s_{i+1} - s_i) \\ y_i + \sin(\theta_i) \cdot (s_{i+1} - s_i) \end{bmatrix} \]

The slide provides a very small velocity at the bottom and the time on the slide is 6.263 s. A longer time on the slide may be obtained by changing the initial angle and the coefficient of friction as well as the length of the slide.
2.70 Solution:

\[ \theta(s) = \left(\frac{s}{25000}\right) \cos\left(\frac{s}{500}\right), \text{ for } 0 < s < 3000 \]

\[ \frac{ds}{dt} = 100 \text{ km/hr} = \frac{100}{3.6} = 27.8 \text{ m/s} \]

\[ \frac{d\theta}{ds} = \frac{1}{25000} \left[ \cos\left(\frac{s}{500}\right) - \frac{s}{500} \cdot \sin\left(\frac{s}{500}\right) \right], N(s) = \cos(\theta(s)) + \frac{v^2}{g} \frac{d\theta}{ds} \]

\[ \Delta s := 1 \quad i := 0 \ldots 3000 \quad s_i := i \cdot \Delta s \]

\[ \theta(s) = s \cdot \frac{\cos\left(\frac{s}{500}\right)}{25000} \]

\[
\begin{pmatrix}
 x_0 \\
 y_0
\end{pmatrix} = \begin{pmatrix}
 0 \\
 0
\end{pmatrix} \quad \begin{pmatrix}
 x_{i+1} \\
 y_{i+1}
\end{pmatrix} = \begin{pmatrix}
 x_i + \cos\left(\theta(s_i)\right) \Delta s \\
 y_i + \sin\left(\theta(s_i)\right) \Delta s
\end{pmatrix}
\]

\[ v := 100 \cdot \frac{1000}{3600} = 27.778 \]

\[ N_i := \cos\left(\theta(s_i)\right) + \frac{v^2}{9.81} \frac{d\theta}{ds} \]

---

\[ N_i \]

---

\[ 1.02 \]

---

\[ x_i \]

---

\[ 1.02 \]

---

\[ 149 \]
2.71 Solution:

\[ s := 0, 10 \ldots 7000 \]
\[ \theta(s) := -\frac{s}{4000} \left( \cos \left( \frac{s}{400} \right) - 1 \right) \]
\[ d\theta(s) := \frac{1}{4000} \left( \cos \left( \frac{s}{400} \right) - \frac{s}{400} \sin \left( \frac{s}{400} \right) \right) - \frac{1}{4000} \]
\[ v := 66 \]
\[ g := 32.2 \]
\[ a_{\text{max}} := 0.5 \cdot g \]
\[ a_n(s) := v \cdot d\theta(s) \]

\[ \begin{array}{cc}
\text{a}_n(s) & \text{a} \_	ext{max} \\
\text{18.941213} & \text{15.547668} \\
\end{array} \]

Yes, the occupant slips near the end of the ride.

2.72 Solution:

\[ 0 < s < 20 \]
\[ \theta = s \sin^{-1}[0.2 - 0.5 \cos(\frac{s}{10})] \]
\[ \dot{s} = 1 - \text{m/s} \]
\[ \sum F_n = F = m \dot{s}^2 \theta' \]
\[ F_{\text{max}} = \mu_s N = \mu_s mg > m \dot{s}^2 \theta' \]
\[ \mu_s g > \dot{s}^2 \theta' \text{ no slip condition} \]
\[ \mu_s > \frac{\dot{s}^2 \theta'}{g} = \frac{10^2 \cdot 0.2571}{9.81} = 2.62 \]
\[ \theta = 5 \sin^{-1}[0.2 - 0.5 \cos(\frac{s}{10})] \]
\[ \theta' = 5 \sqrt{1 - u^2} u' \]
where \( u = 0.2 - 0.5 \cos(\frac{s}{10}) \)
\[ u' = 0.5 \sin(\frac{s}{10}) \cdot \frac{1}{10} \]
\[ \max \theta'(s) = 0.2571 \text{ r/m} \]
\[ @ s = 17.17 \text{ m} \]
The following is a Mathcad analysis:

\[ v := 10 \quad s := 0..20 \]
\[ \theta(s) := 5\cdot\sin\left(0.2 - 0.5\cdot\cos\left(\frac{s}{10}\right)\right) \]
\[ d\theta(s) := \frac{d\theta(s)}{ds} \]
\[ a_n(s) := \frac{v^2}{d\theta(s)} \]

\[ \mu(s) := \frac{a_n(s)}{9.81} \]
\[ \mu(17) = 2.62 \]

2.73 Differentiate the given \( r \) and \( \theta \):
\[
\begin{align*}
r(t) &= 0.300 + 0.100 \cos(\pi t) \\
\dot{r}(t) &= -0.100\pi \sin(\pi t) \\
\ddot{r}(t) &= -0.100\pi^2 \cos(\pi t) \\
\theta(t) &= \frac{\pi}{6} \sin(\pi t) \\
\dot{\theta}(t) &= \frac{\pi^2}{6} \cos(\pi t) \\
\ddot{\theta}(t) &= -\frac{\pi^3}{6} \sin(\pi t)
\end{align*}
\]
\[
\begin{align*}
v(t) &= -0.100\pi \sin(\pi t)\dot{e}_r + (0.300 + 0.100 \cos(\pi t))\frac{\pi^2}{6} \cos^2(\pi t)\dot{e}_\theta \\
a(t) &= \left[-0.100\pi^2 \cos(\pi t) - (0.3 + 0.1 \cos(\pi t))\frac{\pi^4}{36} \cos^2(\pi t)\right]\dot{e}_r \\
&+ [(0.3 + 0.1 \cos(\pi t))(-\frac{\pi^3}{6} \sin(\pi t)) + 2(-0.1\pi \sin(\pi t))(\frac{\pi^2}{6} \cos(\pi t))]\dot{e}_\theta \\
F &= ma_r\dot{e}_r + ma_\theta\dot{e}_\theta \\
&= m\left\{-0.100\frac{\pi^4}{36} \cos^3(\pi t) - 0.300\frac{\pi^4}{36} \cos^2(\pi t) - 0.100\pi^2 \cos(\pi t)\right\}\dot{e}_r \\
&- 0.3m\frac{\pi^3}{6} \sin(\pi t)\{1 + \cos(\pi t)\}\dot{e}_\theta
\end{align*}
\]
2.74 It is given that:

\[ \mathbf{F}(t) = (3t^2 - 1)\hat{e}_r + \cos\left(\frac{\pi t}{6}\right)\hat{e}_\theta, \quad m = 1 \]

Therefore upon comparison with eq. (2.39) in the book:

\[ m(\ddot{r} - r\dot{\theta}^2) = 3t^2 - 1 \]
\[ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \cos\left(\frac{\pi t}{6}\right) \]

This gives the following differential equation to solve:

\[ \ddot{r} = r\dot{\theta}^2 + \frac{1}{m}(3t^2 - 1) \]
\[ \ddot{\theta} = -2\frac{\dot{r}}{r}\dot{\theta} + \frac{1}{mr}\cos\left(\frac{\pi t}{6}\right) \]

With initial conditions

\[ r(0) = 2 \quad \theta(0) = 0 \quad \dot{r}(0) = 0 \quad \dot{\theta}(0) = 0 \]

The solution in MATLAB is given in the following file:

function yprime = ds2pt74(t,y)
\[
m = 1;
Fr = 3*t^2-1;
Fth = cos(t*pi/6);
r = y(1); \quad \text{radial position}
th = y(2); \quad \text{angular position}
v = y(3); \quad \text{radial velocity}
w = y(4); \quad \text{angular velocity}
yprime(1) = v;
yprime(2) = w;
yprime(3) = Fr/m + r*w^2;
yprime(4) = (1/r)*(Fth/m-2*v*w);
return\]
The Mathcad solution follows:

\[
\begin{align*}
    m &:= 1 \\
    ddr(r, \theta, t) &:= r d\theta^2 + \frac{1}{m} \left(3 t^2 - 1\right) \\
    d\theta(dr, r, d\theta, t) &:= -\frac{2}{r} dr d\theta - \frac{1}{mr} \cos \left(\frac{\pi t}{6}\right) \\
    i &:= 0..2000 \quad \Delta t := 0.001 \quad t_i := i \Delta t \\
    \begin{pmatrix}
        dr_0 \\
        r_0 \\
        d\theta_0 \\
        \theta_0
    \end{pmatrix} &:= \begin{pmatrix}
        0 \\
        2 \\
        0 \\
        0
    \end{pmatrix} \quad \begin{pmatrix}
        dr_{i+1} \\
        r_{i+1} \\
        d\theta_{i+1} \\
        \theta_{i+1}
    \end{pmatrix} &:= \begin{pmatrix}
        dr_i + ddr(i, d\theta_i, t_i) \Delta t \\
        r_i + dr_i \Delta t \\
        d\theta_i + d\theta(dr_i, r_i, d\theta_i, t_i) \Delta t \\
        \theta_i + d\theta_i \Delta t
    \end{pmatrix} \\
    x_i &:= r_i \cos(\theta_i) \quad y_i := r_i \sin(\theta_i)
\end{align*}
\]

2.75 The force acting on \(m\) along \(r\) is:

Differentiating the given value of \(\theta\) yields:

\[
F_r = -k(r - 0.2) \hat{e}_r.
\]

\[
\begin{align*}
    \dot{\theta} &= 0.1 t^2 \\
    \ddot{\theta} &= 0.2 t \\
    \dot{\theta} &= 0.2
\end{align*}
\]

Using equation 2.39 yields:

\[
\begin{align*}
    m(\ddot{r} - r \ddot{\theta}^2) &= -k(r - 0.2) \\
    m(r \ddot{\theta} + 2\dot{r} \dot{\theta}) &= 0
\end{align*}
\]

The governing differential equations and initial conditions are:

\[
\begin{align*}
    \ddot{\theta} &= -\frac{2}{4} \hat{\theta} \\
    \ddot{\theta} &= -\frac{2}{4} \hat{\theta} \\
    k &= 400 N/m \quad m = 4 kg \\
    \dot{\theta}(0) &= 0 \quad \dot{r}(0) = 0 \quad r(0) = 0.2 \quad \theta(0) = 0
\end{align*}
\]
The MATLAB code is:

```matlab
function yprime = ds2pt75(t,y)
    m = 4;
    k = 40;
    L = 0.2;
    th = 0.1*t^2;
    thdot = 0.2*t;
    yprime(1) = y(2);
    yprime(2) = y(1)*thdot^2-(k/m)*(y(1)-L);
    return
```

The Mathcad solution is:

```
i := 0..10000
Δt := 0.001
k := 40
m := 4
$t_i := i \cdot Δt$
$θ_i := 0.1 \cdot (t_i)^2$
d$θ_i := 0.2 \cdot t_i$

$ddr(r, dθ) := r \cdot (dθ)^2 - \frac{k}{m} (r - 0.2)$

$\begin{bmatrix}
dr_0
r_0
\end{bmatrix} := \begin{bmatrix}
0
0.2
\end{bmatrix}$

$\begin{bmatrix}
dr_i + 1
r_i + 1
\end{bmatrix} := \begin{bmatrix}
dr_i + ddr(r_i, dθ_i) \cdot Δt
r_i + dr_i \cdot Δt
\end{bmatrix}$
```

Trajectory Plot

Starting point
2.76 Solution:

\[ v = (\dot{r}\hat{e}_x + r\dot{\theta}\hat{e}_\theta) \]

The unit vector along \( v \) for computing \( f \) is:

\[ \frac{v}{|v|} = \frac{\dot{r}\hat{e}_x + r\dot{\theta}\hat{e}_\theta}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}} \]

Equation (2.39) from the book yields:

\[ m (\ddot{r} - r\dot{\theta}^2) = -k(r - 0.2) - \mu_r mg \frac{\dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}} \]

or

\[ \ddot{r} = r\dot{\theta}^2 - \frac{k}{m}(r - 0.2) - \mu_k \cdot \frac{g \cdot \dot{r}}{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}} \]

The MATLAB code is:

```matlab
function yprimem = ds2pt75(t,y) m = 4; k = 20; l = 0.2; g = 9.81; mu = 0.2; th = 0.1*t^2; thdot = 0.2*t; v = sqrt(y(2)^2+(y(1)*thdot)^2);
if v==0,
yprime(1) = y(2);
yprime(2) = y(1)*thdot^2-(k/m)*(y(1)-1);
else
yprime(1) = y(2);
yprime(2) = y(1)*thdot^2-(k/m)*(y(1)-1)-mu*g*y(2)/v;
end
return
```

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The Mathcad solution is:

\[ k := 20\]
\[ m := 4\]
\[ \mu := 0.2\]
\[ g := 9.81\]
\[ \Delta t := 0.001\]
\[ t_i := i \cdot \Delta t\]
\[ \theta (t) := 0.1 \cdot t^2\]
\[ d\theta (t) := 0.2 \cdot t\]
\[ ddr(r, dr, d\theta, t) := r \cdot d\theta^2 - \frac{k}{m} \cdot (r - 0.2) - \mu \cdot g \cdot \frac{dr}{\sqrt{dr^2 + (r \cdot d\theta)^2}}\]

\[
\begin{bmatrix}
  dr_0 \\
  r_0
\end{bmatrix}
:=
\begin{bmatrix}
  0 \\
  0.2
\end{bmatrix}
\]
\[
\begin{bmatrix}
  dr_{i+1} \\
  r_{i+1}
\end{bmatrix}
:=
\begin{bmatrix}
  dr_i + ddr(r_i, dr_i, d\theta(t_i), t_i) \cdot \Delta t \\
  r_i + dr_i \cdot \Delta t
\end{bmatrix}
\]

2.77 We are given \( m = 4 \text{ kg} \), \( k = 650 \text{ N/m} \) and \( \dot{\theta} = 1.5 \text{ rad/s} \)

Differentiate:
\[ r(\theta) = 0.200(2 - \cos \theta)\]
\[ \dot{r}(\theta) = 0.200(\sin \theta)\dot{\theta}\]
\[ \ddot{r}(\theta) = 0.200[(\cos \theta)\ddot{\theta}^2 + (\sin \theta)\dddot{\theta}]\]
The equation of motion in the radial direction is:

\[ N_r - k(r - 0.100) = m(\ddot{r} - r\dot{\theta}^2) \], where \( \dot{\theta} = 1.5 \). Thus:

\[ N_r = k[0.200(2 - \cos \theta) - 0.100] + m[0.200 \cos \theta(1.5)^2 - 0.200(2 - \cos \theta)(1.5)^2] \]

\[ = 650(0.300 - 0.200 \cos \theta) + 4[0.400 \cos \theta(1.5)^2 - 0.200(2)(1.5)^2] \]

Thus

\[ N_r = 191.4 - 126.4 \cos \theta. \]

2.78 We are given \( \dot{\theta} = 2 - \cos \theta \), so that

\[ \ddot{\theta} = \sin \theta \dot{\theta} = \sin \theta(2 - \cos \theta) \]

Thus

\[ N_r = k(r - 0.100) + m(\ddot{r} - r\dot{\theta}^2) \]

\[ \ddot{r}(t) = 0.200[\cos \theta(2 - \cos \theta)^2 + \sin^2 \theta(2 - \cos \theta)] \]

\[ r\dot{\theta}^2 = 0.200(2 - \cos \theta)(2 - \cos \theta)^2 \]

The following MATLAB code computes the solution

```matlab
function Nr = ds2pt78(theta)
    thdot = 2-cos(theta);
    thddot = sin(theta).*thdot;
    r = 0.2*(2-cos(theta));
    rdot = 0.2*sin(theta).*thdot;
    rddot = 0.2*sin(theta).*thddot + 0.2*cos(theta).*thdot.^2;
    k = 650; m = 4; l = 0.1;
    Nr = m*(rddot-r.*thdot.^2)+k*(r-l);
    return
```

The following computes the solution in Mathcad:
\[ \theta := 0, \frac{\pi}{48} \ldots 4\pi \]

\[ F_s(\theta) := 650\cdot[0.2 \cdot (2 - \cos(\theta)) - 0.1] \]

\[ ddr(\theta) := 0.2\cdot[\cos(\theta) \cdot (2 - \cos(\theta))^2 + (\sin(\theta))^2 \cdot (2 - \cos(\theta))] \]

\[ r\cdot\omega(\theta) := 0.2 \cdot (2 - \cos(\theta))^3 \]

\[ N_t(\theta) := F_s(\theta) + 4 \cdot (ddr(\theta) - r\cdot\omega(\theta)) \]

2.79 Following the solution of sample Problem 2.17 the following codes are used to numerically adjust the mass until the desired response results. In MATLAB, the code is:

```matlab
function yprime = ds2pt79(t,y)
m = 10.6;
k = 500;
r0 = 0.3;
g = 9.81;
r = y(1);
theta = y(2);
rdot = y(3);
thdot = y(4);
% velocities
yprime(1,:) = rdot;
yprime(2,:) = thdot;
% accelerations
yprime(3,:) = r*thdot^2-(k/m)*(r-r0)+g*cos(theta);
yprime(4,:) = (-2*rdot*thdot-g*sin(theta))/r;
return
```

In Mathcad the code is:
The mass was adjusted by trial and error to produce the required path of motion. 

\[ a(v, r, \omega, \theta) := \left( r \cdot \omega^2 + g \cdot \cos(\theta) \right) - \frac{k}{m} (r - 0.3) \]

\[ \alpha(v, r, \omega, \theta) := -\frac{1}{r} \left( g \cdot \sin(\theta) + 2 \cdot v \cdot \omega \right) \]

\[
\begin{bmatrix}
   v_{i+1} \\
   r_{i+1} \\
   \omega_{i+1} \\
   \theta_{i+1}
\end{bmatrix} :=
\begin{bmatrix}
   0 \\
   0.3 \\
   0 \\
   30\,\text{deg}
\end{bmatrix}
\]

\[ m := 3.6 \text{ kg} \]

Pendulum Trajectory

vertical position (m)

horizontal position (m)

2.80 Solution:
FIGURE S2.80

From the free body diagram, the equations of motion can be written:

\[-k(r - L) + mg \cos \theta = m\ddot{r} - mr\dot{\theta}^2\]

\[\ddot{r} = r\dot{\theta}^2 + g \cos \theta - \frac{k}{m}(r - L)\]

and:

\[-mg \sin \theta = 2m\dot{r}\dot{\theta} + mr\ddot{\theta}\]

\[\dot{\theta} = -\frac{2}{r} \sin \theta - 2\frac{\dot{r}}{r}\dot{\theta}\]

The MATLAB code to solve the problem numerically is:

```matlab
function yprime = ds2pt80(t,y)
m = 5;
r = y(1);
th = y(2);
rdot = y(3);
 thdot = y(4);
Fr = 1-sin(th);
Fth = -(2+t-t^2);
yprime(1) = rdot;
yprime(2) = thdot;
yprime(3) = r*thdot^2+Fr/m;
yprime(4) = (-2*rdot*thdot+Fth/m)/r;
return
```

The numerical solution in Mathcad is given on the following page. Note that the solution (pendulum angle versus time) to Sample Problem 2.13 is also computed and plotted (in blue). The two solutions differ when \(k = 60\) but are in good agreement when \(k = 600\). The length of the pendulum is also plotted as a function of time showing the degree to which the mass is oscillating on the spring (note that the oscillations are very small when \(k = 600\), i.e. in that case the pendulum is behaving approximately as if the spring were a rigid bar or string).
\[ m := 2 \quad L := 2 \quad g := 9.81 \]

\[ \ddot{r}(r, \dot{\theta}, \theta, k) := r \dot{\theta}^2 + g \cos(\theta) - \frac{k}{m} (r - L) \]

\[ \ddot{\theta}(r, \dot{r}, \theta, \dot{\theta}) := \frac{-g}{r} \sin(\theta) - 2 \frac{\ddot{r}}{r} \]

\[ i := 0 \ldots 60000 \quad \Delta t := 0.0001 \quad t_i := i \cdot \Delta t \]

\[
\begin{pmatrix}
dr_0 \\
r_0 \\
d\theta_0 \\
\theta_0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
L \\
0 \\
30\text{-deg}
\end{pmatrix}
\begin{pmatrix}
dr_{i+1} \\
r_{i+1} \\
d\theta_{i+1} \\
\theta_{i+1}
\end{pmatrix}
= 
\begin{pmatrix}
\dot{r}_i + d\dot{r}(r_i, \dot{\theta}_i, \theta_i, 60) \Delta t \\
r_i + \dot{r}_i \Delta t \\
d\theta_i + d\theta(r_i, \dot{r}_i, \dot{\theta}_i) \Delta t \\
\theta_i + d\theta_i \Delta t
\end{pmatrix}
\]

\[
\begin{pmatrix}
d\beta_0 \\
\beta_0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
30\text{-deg}
\end{pmatrix}
\begin{pmatrix}
d\beta_{i+1} \\
\beta_{i+1}
\end{pmatrix}
= 
\begin{pmatrix}
d\beta_i - \frac{g}{L} \sin(\beta_i) \Delta t \\
\beta_i + d\beta_i \Delta t
\end{pmatrix}
\]
2.81 The free-body diagram is:

![Diagram](image1.png)

FIGURE S2.81

Using equation (2.39) in the book the equations of motion are:

\[-N + mg \cos \theta = m(\ddot{r} - r\dot{\theta}^2)\]

\[-mg \sin \theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\]

Subject to: \(\dot{\theta}(0) = \frac{v_0}{R}\)

we have that: \(\ddot{\theta} = \frac{\omega_0}{d\omega} = -\frac{g}{R} \sin \theta\), or: \(\int_{\omega(0)}^{\omega} \omega d\omega = \int_0^\theta -\frac{g}{R} \sin \theta d\theta\)

\(\frac{1}{2}(\omega^2 - \frac{v_0^2}{R^2}) = \frac{g}{R}(\cos \theta - 1)\), thus: \(\omega^2 = \frac{2g}{R}(\cos \theta - 1) + \frac{v_0^2}{R^2}\)

The normal force will be a minimum at \(\theta = 180^\circ\)

\(N = -mg + mR\omega^2\)

Therefore \(\omega^2 \geq \frac{g}{R}(-2) + \frac{v_0^2}{R^2}\)

\(v_0 = \sqrt{5gR}\)

2.82 The free-body diagram is:

![Diagram](image2.png)

FIGURE S2.82

From the diagram and equation (2.39) in the book:

\[-N + mg \cos \theta = -mR\dot{\theta}^2\]
\[-mg \sin \theta - \mu_k |N| \frac{\dot{\theta}}{|\dot{\theta}|} = mR\ddot{\theta}\]

Solving for the normal force:

\[N = mg \cos \theta + mR\dot{\theta}^2\]

And substituting this into the equation for angular acceleration:

\[\ddot{\theta} = -\frac{R}{g} \sin \theta - \frac{\mu_k}{R^2} |g \cos \theta + R\dot{\theta}^2| \frac{\dot{\theta}}{|\dot{\theta}|}\]

The computer solution in Mathhead is (independent of mass):

\[
\begin{align*}
R &:= 2 \quad \mu_k := 0.2 \quad v_0 := 5 \quad g := 9.81 \\
\alpha(\omega, \theta) &:= -\frac{g}{R} \sin(\theta) - \frac{\mu_k}{R} \left| g \cos(\theta) + R \omega^2 \right| \frac{\omega}{|\omega|} \\
i &:= 0 .. 5000 \quad \Delta t := 0.001 \quad t_i := i \Delta t \\
\begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix} &:= \begin{pmatrix} v_0 \\ R \end{pmatrix} \quad \begin{pmatrix} \omega_{i+1} \\ \theta_{i+1} \end{pmatrix} := \begin{pmatrix} \omega_i + \alpha(\omega_i, \theta_i) \Delta t \\ \theta_i + \omega_i \Delta t \end{pmatrix}
\end{align*}
\]

\[\text{FIGURE S2.83a}\]

\[L^2 = R^2 + \frac{R^2}{4} - R^2 \cos(\theta)\]

\[L = \frac{R}{2} \sqrt{5 - 4 \cos \theta}\]
And:
\[
\frac{R^2}{4} = R^2 + L^2 - 2RL \cos \beta
\]
\[
\frac{R^2}{4} = R^2 + \frac{R^2}{4} (5 - 4 \cos \theta) - R^2 \sqrt{5 - 4 \cos \theta} \cos \beta
\]
\[
\cos \beta = \frac{2 - \cos \theta}{\sqrt{5 - 4 \cos \theta}}
\]
From the law of sines:
\[
\frac{\sin \theta}{L} = \frac{\sin \beta}{R/2}
\]
\[
\sin \beta = \frac{\sin \theta}{\sqrt{5 - 4 \cos \theta}}
\]
The spring force is:
\[
F_s = k(L - L_0) = kR \left[ \sqrt{5 - 4 \cos \theta} - \frac{1}{4} \right]
\]
Components of the spring force are:
\[
F_{sn} = -F_s \cos \beta = -\frac{kR}{2} \left[ 1 - \frac{1}{4\sqrt{5 - 4 \cos \theta}} \right] (2 - \cos \theta)
\]
\[
F_{s\theta} = -F_s \sin \beta = -\frac{kR}{2} \left[ 1 - \frac{1}{4\sqrt{5 - 4 \cos \theta}} \right] \sin \theta
\]
From the free-body diagram in S2.83b, the equation of motion is:
\[
mR^2 \frac{d^2 \theta}{dt^2} = -kR \left[ 1 - \frac{1}{4\sqrt{5 - 4 \cos \theta}} \right] \sin \theta
\]
\[
\frac{d^2 \theta}{dt^2} = -\frac{k}{2m} \left[ 1 - \frac{1}{4\sqrt{5 - 4 \cos \theta}} \right] \sin \theta
\]
2.84 Solution:

With the addition of the weight as shown in S2.84, the equation of motion becomes:

\[ mR \ddot{\theta} = - \frac{kR}{2} \left[ 1 - \frac{1}{4\sqrt{5} - 4\cos \theta} \right] \sin \theta - mg \cos \theta \]

\[ \frac{d^2 \theta}{dt^2} = - \frac{k}{2m} \left[ 1 - \frac{1}{4\sqrt{5} - 4\cos \theta} \right] \sin \theta - \frac{\mu_k N}{kR} \dot{\theta} \]

2.85 Solution:

The friction force is:

\[ f = -\mu_k N \frac{\dot{\theta}}{|\dot{\theta}|} \]

With the addition of the friction force as shown in S2.85, the equation of motion becomes:

\[ mR \ddot{\theta} = - \frac{kR}{2} \left[ 1 - \frac{1}{4\sqrt{5} - 4\cos \theta} \right] \sin \theta - \mu_k N \frac{\dot{\theta}}{|\dot{\theta}|} \]

\[ \frac{d^2 \theta}{dt^2} = - \frac{k}{2m} \left[ 1 - \frac{1}{4\sqrt{5} - 4\cos \theta} \right] \sin \theta - \frac{\mu_k N}{mR} \frac{\dot{\theta}}{|\dot{\theta}|} \]

where:

\[ N = \frac{kR}{2} \left[ 1 - \frac{1}{4\sqrt{5} - 4\cos \theta} \right] (2 - \cos \theta) - mR \dot{\theta}^2 \]

---

FIGURE S2.84

FIGURE S2.85
With the addition of both the weight and the friction force as shown in S2.86, the equation of motion becomes:

\[
m R \frac{d^2 \theta}{dt^2} = -mg \cos \theta - \frac{kR}{2} \left[1 - \frac{1}{4 \sqrt{5 - 4 \cos \theta}}\right] \sin \theta - \mu_k N \frac{\dot{\theta}}{|\dot{\theta}|}
\]

\[
\frac{d^2 \theta}{dt^2} = -\frac{g}{R} \cos \theta - \frac{k}{2m} \left[1 - \frac{1}{4 \sqrt{5 - 4 \cos \theta}}\right] \sin \theta - \frac{\mu_k N \dot{\theta}}{mR |\dot{\theta}|}
\]

where:

\[
N = mg \sin \theta + \frac{kR}{2} \left[1 - \frac{1}{4 \sqrt{5 - 4 \cos \theta}}\right] (2 - \cos \theta) - mR \dot{\theta}^2
\]

A Mathcad code for integrating the equation of motion is as follows with some choice of \(m, k, R, \mu_k\) and \(v_0\) such the the motion damps out after two oscillations. There are infinitely many other combinations that would satisfy this criterion.
2.87 We are given:

\[ M = 5.98 \times 10^{24} kg \]
\[ r = 6380 \times 10^3 \text{ m} \]
\[ G = 66.7 \times 10^{-12} \text{ m}^3/\text{kg} \cdot \text{s}^2 \]

\[ \frac{GmM}{r^2} = \frac{m v^2}{r} \]

\[ v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{5.98 \times 66.7}{500+6380} \times 10^3} = 7614 \text{ m/s} \]
2.88 Solution:
\[ R \omega^2 m = \frac{GMm}{R^2} \]
\[ \omega = \sqrt{\frac{GM}{R^3}} \]
The period \( \tau \) is
\[ \tau = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{GM}} R^3 \]

2.89 Solution:
\[ \tau = 23 \text{ hrs 56 min. } = 86.16 \times 10^3 \text{ s.} \]
\[ \tau = \frac{2\pi}{\sqrt{GM}} R^{3/2} \]
\[ r_0^{3/2} = \frac{\tau \sqrt{GM}}{2\pi} = 273.87 \times 10^9 \text{ m} \]
\[ r_0 = 42210 \text{ km.} \]
\[ r = r_0 - R = 42210 - 6380 \]
\[ r = 35830 \text{ km} \]
\[ v = \sqrt{\frac{GM}{r_0}} = 3074 \text{ km/s.} \]

2.90 Solution:
First, let us determine the velocity at final circular orbit of 700 km altitude:
\[ R_4 = 6380 + 700 = 7080 \text{ km} \]
\[ v_4 = \sqrt{\frac{GM}{R_4}} = 7.506 \times 10^3 \text{ m/s} \]
\[ R_1 = 6380 + 400 = 6780 \text{ km} \]
\[ v_1 = \sqrt{\frac{GM}{R_1}} = 7.670 \times 10^3 \text{ m/s} \]
From Eq. (2.67) for a conic section:
\[ \frac{1}{r} = A(1 + e \cos \theta) \]
For elliptic orbit \( r_0 \Rightarrow \theta = 0, \ r_{\text{apogee}} \Rightarrow \theta = 180 \)
Therefore \[ r_{\text{apogee}} = \frac{r_0}{r_0} = \frac{(1+e)}{(1-e)} \]
Solving for \( e \): \[ e = \frac{(r_a/r_0) - 1}{(r_a/r_0) + 1} \]
\[ \frac{r_a}{r_0} = \frac{7080}{6780} = 1.044 \]
\[ e = 0.022 \]
\( h \) is the angular momentum/unit mass = \( r_0 v_0 \). From eq. 2.69
\[ e = \frac{h^2}{GMr_0} - 1 = \frac{r_0 v_0^2}{GM} - 1 \]
The velocity to go into elliptical orbit is
\[ v_2 = \sqrt{\frac{GM}{R_1}} (1 + e) = 7.754 \times 10^3 \text{ m/s} \]
Since angular momentum is concerned in elliptical orbit
\[ v_3 r_a = v_2 r_0, \text{ or } v_3 = v_2 \frac{r_0}{r_a} = 7.411 \times 10^3 \text{ m/s} \]

Therefore the first boost \( v_1 \rightarrow v_2 \) is \( 7.670 \times 10^3 \text{ m/s} \) or \( 7.754 \times 10^3 \text{ m/s} \)

The second boost \( v_3 \rightarrow v_4 \) is
\[ 7.411 \times 10^3 \rightarrow 7.506 \times 10^2 \text{ m/s}. \]

2.91 From Problem 2.89
\[ R_4 = 42,170 \text{ km} \]
\[ v_4 = 3.075 \times 10^3 \text{ m/s} \]

From Problem 2.90
\[ R_1 = 6780 \text{ km}, \]
\[ v_1 = 7.670 \times 10^3 \text{ m/s} \]
\[ e = \frac{R_4/R_1 - 1}{R_4/R_1 + 1} = 0.723 \]
\[ v_2 = \sqrt{\frac{GM}{R_1}} (1 + e) = 10.070 \times 10^3 \text{ m/s} \]
\[ v_3 = \frac{R_4}{R_1} v_2 = 1.619 \times 10^3 \text{ m/s} \]

Therefore \( v_1 \rightarrow v_2: 7.670 \times 10^3 \rightarrow 10.070 \times 10^3 \text{ m/s} \)
\[ v_3 \rightarrow v_4: 1.619 \times 10^3 \rightarrow 3.075 \times 10^3 \text{ m/s} \]

2.92 For an elliptical orbit
\[ e = 0.12 \quad v_p = 4000 \text{ m/s} \]
\[ \frac{1}{r_0} = \frac{GM}{r_0^2} [1 + e], \quad h = r_0 v_0 \]
\[ r_0 = \frac{GM}{v_0^2} (1 + e), \quad M = 5.98 \times 10^{24} \text{ kg}, \quad G = 66.7 \times 10^{-8} \text{ m}^2/\text{kg} \cdot \text{s}^2 \]
\[ r_0 = \frac{5.98 \times 66.7 \times 10^{12}}{16 \times 10^6} (1 + 0.12) = 27.92 \times 10^6 \text{ m} \]

Therefore \( k = 111.68 \times 10^9 \text{ m}^2/\text{s} \)

At the apogee
\[ \frac{1}{r_a} = \frac{GM}{r_a^2} (1 - e) \]
\[ \frac{1}{r_a} = \frac{5.98 \times 66.7 \times 10^2}{111.68 \times 10^9} (0.88) = 2.8 \times 10^{-8} \]
\[ r_a = 35.7 \times 10^6 \text{ m} \quad v_a = \frac{k}{r_a} = 3128 \text{ m/s} \]

Altitude = \( r_a - r_e = (35.7 - 6.378) \times 10^6 = 29,332 \text{ km} \)
2.93 Solution:

For a parabolic arc: $e = 1$

\[
\frac{1}{\tau} = \frac{GM}{r^2} (1 + e \cos \theta)
\]

$r = 200,000,000$, $\theta = 140^\circ$

\[
\cos 140^\circ = -0.766
\]

Therefore $h^2 = 2 \times 10^8 \times 5.98 \times 10^{24} \times 66.7 \times 10^{-12} (1 - 0.766)$

\[
h = 136.6 \times 10^9
\]

\[
d = \frac{h}{v} = \frac{136.6 \times 10^9}{2.222 \times 10^4}
\]

\[
d = 23,395 \text{ km}
\]

$R_e = 6380 \text{ km}$

Altitude = $d - R_e = 17,015 \text{ km}$

2.94 Solution:

$r_a = 6380 \times 10^3 + 392.62 \times 10^3 = 6772.62 \times 10^3 \text{ m}$

$r_p = 6380 \times 10^3 + 385.42 \times 10^3 = 6765.42 \times 10^3 \text{ m}$

\[
\frac{1}{r_p} = \frac{GM}{r^2} [1 + e], \quad e = \frac{h^2}{GMr_0} - 1
\]

\[
\tau = 92.34 \cdot 60 = 5.54 \times 10^3 \text{ seconds}
\]

\[
h = \frac{\pi}{\tau} (r_a + r_p) \sqrt{r_a r_p} = 5.196 \times 10^{10}
\]

\[
e = \frac{h^2}{GMr_p} - 1 = 5.108 \times 10^{-4}
\]

2.95 Final velocity (Mir’s velocity at the perigee), using values from 2.94:

\[
v_4 = \sqrt{\frac{GM}{r_p} (1 + e)} = 7.687 \times 10^3 \text{ m/s}
\]

$R_1 = 6380 + 100 = 6480 \text{ km}$

\[
v_1 = \sqrt{\frac{GM}{R_1}} = \sqrt{\frac{5.98 \times 66.7}{6.480} \times 10^3} = 7.846 \times 10^3 \text{ m/s}
\]
The Hohmann transfer-elliptic intercept orbit

\[ R_1 = 6480 \text{ km}, \ r_p = 6765 \text{ km} \]

\[ e_2 = \frac{r_p}{r_a+1} = 0.022 \]

The velocity to go into this orbit is

\[ v_3 = \sqrt{\frac{GM}{R_1}}(1 + e_2) = 7932 \times 10^3 \text{ m/s} \]

Conservation of momentum:

\[ v_3 = 7932 \times 10^3 \times \frac{R_1}{r_p} = 7598 \times 10^3 \text{ m/s} \]

\[ v_1 \rightarrow v_2 \quad 7.846 \rightarrow 7.932 \text{ km/s} \]

\[ v_3 \rightarrow v_4 \quad 7.598 \rightarrow 7.687 \text{ km/s} \]

2.96 Solution: \( v_4 = \sqrt{\frac{MG}{R_4}}, \ R_4 = 6380 + 150 = 6530 \text{ km} \)

\[ v_4 = 7816 \text{ m/s} \]

\( R_1 = 6765 \text{ km} \quad v_1 = 7687 \text{ m/s} \)

\[ e = \frac{4u - 1}{r_p + 1}, \ r_a = 6765, \ r_p = 6530 \quad \frac{r_a}{r_p} = 1.036, \]

\[ e = 0.018, \]

v<sub>2</sub> to go into elliptic transfer orbit

\[ v_2 = \sqrt{\frac{GM}{R}}(1 - e) = 7618 \text{ m/s} \]

\[ v_3 = 7618 \times \frac{r_a}{r_p} = 7892 \text{ m/s} \]

Retrofiring

\[ v_1 \rightarrow v_2 \quad 7687 \rightarrow 7618 \text{ m/s} \]

\[ v_3 \rightarrow v_4 \quad 7892 \rightarrow 7816 \text{ m/s} \]
2.97 Solution:

The equations of motion are

\[ -N_\omega = -mr \ddot{\theta}^2 \]
\[ -\mu N_\omega \cos \beta + N_b \sin \beta - \mu N_b \cos \beta = mr \ddot{\theta} \]
\[ -mg + \mu N_\omega \sin \beta + N_b \cos \beta + \mu N_b \sin \beta = m \ddot{z} \]

Constraint: \( a = a_{\text{bu}}(+ \cos \beta \hat{e}_\theta - \sin \beta \hat{k}) - r \ddot{\theta}^2 \hat{e}_r \)

\[ -\mu mr \ddot{\theta}^2 \cos \beta + N_b (\sin \beta - \mu \cos \beta) = ma \cos \beta \quad (1) \]
\[ -mg + \mu mk \ddot{\theta}^2 \sin \beta + N_b (\cos \beta + \mu \sin \beta) = -ma \sin \beta \quad (2) \]

Mulp (1) by \( \sin \beta \) and (2) by \( \cos \beta \) and add.

\[ -mg \cos \beta + N_b (\cos^2 \beta - \mu \sin \beta \cos \beta + \sin^2 \beta - \mu \sin \beta \cos \beta) = 0 \]

Therefore \( N_b = \frac{mg \cos \beta}{1 - 2\mu \sin \beta \cos \beta} \)

\[ a = \frac{1}{m} (-\mu mr \ddot{\theta}^2 + \frac{mg (\sin \beta - \mu \cos \beta)}{1 - 2\mu \sin \beta \cos \beta}) \]

Therefore

\[ \ddot{\theta} = \alpha = \frac{1}{mr} \left[ -\mu mr \ddot{\theta}^2 \cos \beta + \frac{mg \cos \beta (\sin \beta - \mu \cos \beta)}{1 - 2\mu \sin \theta \cos \beta} \right] \]
\[ \ddot{z} = \frac{1}{m} \left[ -mg + \mu mr \ddot{\theta}^2 \sin \beta + \frac{mg \cos \beta (\cos \beta + \mu \sin \beta)}{1 - 2\mu \sin \beta \cos \beta} \right] \]
\[ \ddot{z} = +\mu r \ddot{\theta}^2 \sin \beta - \frac{g \sin \beta (\sin \beta - \mu \cos \beta)}{1 - 2\mu \sin \beta \cos \beta} \]

The Mathcad solution:
\[ i := 0 \ldots 5000 \]
\[ \Delta t := 0.001 \]
\[ t_i := i \cdot \Delta t \]
\[ g := 9.81 \]
\[ \beta := 30{\text{ deg}} \]
\[ \mu := 0.1 \]
\[ r := 0.5 \]
\[ \alpha(\omega) := \frac{1}{r} \left[ -\mu \cdot r \cdot |\omega| \cdot \cos(\beta) + g \cdot \cos(\beta) \cdot \frac{(\sin(\beta) - \mu \cdot \cos(\beta))}{1 - 2 \cdot \mu \cdot \sin(\beta) \cdot \cos(\beta)} \right] \]
\[ a(\omega) := \mu \cdot r \cdot |\omega| \cdot \sin(\beta) + g \cdot \sin(\beta) \cdot \frac{(-\sin(\beta) + \mu \cdot \cos(\beta))}{1 - 2 \cdot \mu \cdot \sin(\beta) \cdot \cos(\beta)} \]
\[
\begin{bmatrix}
  v_0 \\
  z_0 \\
  \omega_0 \\
  \theta_0
\end{bmatrix}
:=
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]
\[
\begin{bmatrix}
  v_i + 1 \\
  z_i + 1 \\
  \omega_i + 1 \\
  \theta_i + 1
\end{bmatrix}
:=
\begin{bmatrix}
  v_i + a(\omega_i) \cdot \Delta t \\
  z_i + v_i \cdot \Delta t \\
  \omega_i + \alpha(\omega_i) \cdot \Delta t \\
  \theta_i + \omega_i \cdot \Delta t
\end{bmatrix}
\]
We are given:

\[ v_\theta = r \dot{\theta}, \quad r = 1.5, \quad h = 50, \quad v_{\theta 0} = \frac{20}{1.5} \]

\[ a = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \hat{e}_\theta + \ddot{z} \hat{k} \]

\[ \mathbf{F} = -N \hat{e}_x - mg \hat{k} \]

\[ \dot{r} = \ddot{r} = 0 \]

\[ N = mr\dot{\theta}^2 \]

\[ 0 = r \ddot{\theta} \]

\[ -mg = m \ddot{z} \]

\[ \ddot{\theta} = 0, \quad \dot{\theta} = \dot{\theta}_0, \quad \theta = \dot{\theta}_0 t \]

\[ \ddot{z} = -g, \quad \dot{z} = -gt, \quad z = -\frac{1}{2}gt^2 + z_0 \]

The following MATLAB code plots this:

```matlab
clear all
format short e
format compact
t = linspace(0,2);
g = 9.81;
v0 = 20;
R = 1.5;
w0 = v0/R;
theta = w0*t;
z = -(g/2)*t.^2;
r = R*ones(size(t));
x = r.*cos(theta);
y = r.*sin(theta);
```
plot3(x,y,z)

The Mathcad code is:

\[
i := 0 .. 200
\]

\[
t := \frac{i}{100}
\]

\[
\omega := \frac{20}{1.5}
\]

\[
x := 1.5 \cdot \cos(\omega \cdot t_i)
\]

\[
y := 1.5 \cdot \sin(\omega \cdot t_i)
\]

\[
z_i := -\frac{1}{2} \cdot 9.81 \cdot (t_i)^2 + 50
\]
2.99 Solution:

\[
v = \begin{bmatrix} 0 \\ r \cdot \omega \\ vz \end{bmatrix}
\]

\[
f = -\mu N v / |v|
\]

\[
f = \frac{-\mu mr \dot{\theta}^2}{\sqrt{r^2 \omega^2 + vz^2}} (r \omega \hat{e}_\theta + vz \hat{k})
\]

\[
fr = m \ddot{r} = \frac{-\mu mr \dot{\theta}^2}{\sqrt{r^2 \omega^2 + vz^2}} r \dot{\theta}, \quad \ddot{z} = -g - \frac{\mu r \dot{\theta}^2 vz}{\sqrt{r^2 \omega^2 + vz^2}}
\]

These are solved numerically by the following MATLAB code:

```matlab
function yprime = ds2pt99(t,y)
    R = 1.5;
g = 9.81;
mu = 0.2;

    th = y(1);
z = y(2);
    thdot = y(3);
zdot = y(4);

    v = sqrt(R*thdot^2+zdot^2);
    vprime(1) = thdot;
    yprime(2) = zdot;
    yprime(3) = -mu*R*thdot^3/v;
    yprime(4) = -mu*R*thdot^2*zdot/v-g;

    return
```

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The Mathcad code follows:

\[
i := 0 \ldots 390 \\
\Delta t := 0.01 \\
t := i \cdot \Delta t \\
g := 9.81 \\
\mu := 0.2 \\
r := 1.5 \\
\alpha(vz, \omega) := \frac{-\mu \cdot r \cdot \omega^3}{\sqrt{(r \cdot \omega)^2 + vz^2}} \\
a(vz, \omega) := -g - \frac{\mu \cdot r \cdot \omega^2 \cdot vz}{\sqrt{(r \cdot \omega)^2 + vz^2}} \\
\begin{bmatrix} vz_0 \\ z_0 \\ \omega_0 \\ \theta \end{bmatrix} := \begin{bmatrix} 0 \\ 50 \\ 20 \\ 1.5 \end{bmatrix} \\
\begin{bmatrix} vz_{i+1} \\ z_{i+1} \\ \omega_{i+1} \\ \theta_{i+1} \end{bmatrix} := \begin{bmatrix} vz_i + a(vz, \omega_i) \cdot \Delta t \\ z_i + vz_i \cdot \Delta t \\ \omega_i + \alpha(vz, \omega_i) \cdot \Delta t \\ \theta + \omega_i \cdot \Delta t \end{bmatrix} \\
n := 0 \ldots 391 \\
x := r \cdot \cos(\theta_n) \\
y := r \cdot \sin(\theta_n) \\
z_n := z_n \\
\]

3-D Trajectory
2.100 This follows 2.99 and the codes are the same with different initial condition.

The Mathcad version follows:

\[
\begin{align*}
i &:= 0 \ldots 390 \\
\Delta t &:= 0.01 \\
t &:= i \cdot \Delta t \\
g &:= 9.81 \\
\mu &:= 0.2 \\
r &:= 1.5 \\
\alpha(vz, \omega) &:= \frac{-\mu \cdot r \cdot \omega}{\sqrt{(r \cdot \omega)^2 + vz^2}} \\
a(vz, \omega) &:= -g - \frac{\mu \cdot r \cdot \omega \cdot vz}{\sqrt{(r \cdot \omega)^2 + vz^2}} \\
\begin{bmatrix} vz_0 \\ z_0 \\ \omega_0 \\ \theta_0 \end{bmatrix} &:= \begin{bmatrix} 20 \\ 30 \\ 20 \\ 1.5 \end{bmatrix} \\
\begin{bmatrix} vz_i + 1 \\ z_i + 1 \\ \omega_i + 1 \\ \theta_i + 1 \end{bmatrix} &:= \begin{bmatrix} vz_i + a(vz_i, \omega_i) \cdot \Delta t \\ z_i + vz_i \cdot \Delta t \\ \omega_i + \alpha(vz_i, \omega_i) \cdot \Delta t \\ \theta_i + \omega_i \cdot \Delta t \end{bmatrix} \\
n &:= 0 \ldots 391 \\
x_n &:= r \cdot \cos(\theta_n) \\
y_n &:= r \cdot \sin(\theta_n) \\
z_n &:= z_n
\end{align*}
\]
2.101 We have \( v_\theta = v_0 \cos \theta \) and \( v_z = v_0 \sin \beta \)

\[
\begin{align*}
\mathbf{N} &= -N\hat{e}_R \\
\mathbf{W} &= mg(\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) \\
m\ddot{z} &= 0 \\
-mr\dot{\theta}^2 &= -N + mg \cos \theta \\
mr\dot{\theta} &= -mg \sin \theta
\end{align*}
\]

Eq. (3) may be written
\( \omega_{\theta} \frac{d\omega}{d\theta} = -g/r \sin \theta \)
Integrating yields:
\[
\omega^2 - \omega_0^2 = \frac{2g}{r}(\cos \theta - 1)
\] (4)

From Eq. 2, minimum velocity can be determined when \( \theta = 180^\circ \) and \( N = 0 \). The angular velocity \( \omega_{180}^2 = g/r \). From (4)
\[
g/r - \omega_0^2 = \frac{7g}{r} \quad \text{Therefore } \omega_0 = \sqrt{\frac{5g}{r}}
\]
\[
v_{\theta_0} = \sqrt{4gr} = v_0 \cos \beta, \quad v_0 = \sqrt{\frac{5gr}{\cos \beta}}.
\]

2.102 From the previous problem the forces are
\[
\begin{align*}
\mathbf{N} &= -N\hat{e}_r \\
\mathbf{W} &= mg(\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) \\
\mathbf{f} &= \frac{-\mu N}{\sqrt{(r\omega)^2 + v_z^2}}(r\omega \hat{e}_0 + v_z \hat{e}_z)
\end{align*}
\]
\[
r\omega_0 = v_0 \cos \beta \\
v_{z0} = v_0 \sin \beta \\
-N + mg \cos \theta = -mr\omega^2
\]
\[m\ddot{z} = -\mu N v_z / \sqrt{(r\omega)^2 + v_z^2}\]

\[mr\ddot{\theta} = -mg \sin \theta - \mu N r \omega / \sqrt{(r\omega)^2 + v_z^2}\]

From 2.101 \(\omega = \sqrt{g/r}\) when \(\theta = 180^\circ\)

\[N = mg \cos \theta + mr\omega^2\]

Therefore \(\ddot{z} = -\frac{\mu (g \cos \theta + r\omega^2) v_z}{\sqrt{(r\omega)^2 + v_z^2}}\)

\(\ddot{\theta} = -\frac{z \sin \theta - \mu (g \cos \theta + r\omega^2) \omega}{\sqrt{(r\omega)^2 + (v_z)^2}}\)

The minimum initial velocity of 10.6 m/s was determined by trial and error using the following codes. The motion damps out after 4 seconds and the particle comes to rest at 360 degrees. The MATLAB code is:

```matlab
function yprime = ds2pt102(t,y)
  m = 3;
  R = 0.6;
  g = 9.81;
  mu = 0.2;
  th = y(1);
  z = y(2);
  thdot = y(3);
  zdot = y(4);
  v = sqrt((R*thdot)^2+zdot^2);
  N = m*g*cos(th)+m*R*thdot^2;
  yprime(1) = thdot;
  yprime(2) = zdot;
  yprime(3) = (-m*g*sin(th)-mu*N*R*thdot/v)/m/R;
  yprime(4) = (-mu*N*zdot/v)/m;
  return
```
The Mathcad code is:

\begin{align*}
I &:= 0..5000 \\
\Delta t &:= 0.001 \\
t_i &:= i \cdot \Delta t \\
\mu &:= 0.2 \quad \beta := 50\deg \quad g := 9.81 \quad v_0 := 10.6 \quad r := 0.6 \\
V(vz, \omega) &:= \sqrt{(r \cdot \omega)^2 + vz^2} \\
a(vz, \omega, \theta) &:= -\mu \cdot (g \cdot \cos(\theta) + r \cdot \omega^2) \cdot \frac{vz}{V(vz, \omega)} \\
\alpha(vz, \omega, \theta) &:= \frac{-g}{r} \sin(\theta) - \mu \cdot (g \cdot \cos(\theta) + r \cdot \omega^2) \cdot \frac{\omega}{V(vz, \omega)} \\
\begin{bmatrix}
vz_0 \\
z_0 \\
\omega_0 \\
\theta_0
\end{bmatrix} &:= \\
\begin{bmatrix}
v_0 \cdot \sin(\beta) \\
0 \\
v_0 \cdot \frac{\cos(\beta)}{r} \\
0
\end{bmatrix} \\
\begin{bmatrix}
vz_{i+1} \\
z_{i+1} \\
\omega_{i+1} \\
\theta_{i+1}
\end{bmatrix} &:= \\
\begin{bmatrix}
vz_i + a(vz_i, \omega_i, \theta_i) \cdot \Delta t \\
z_i + vz_i \cdot \Delta t \\
\omega_i + \alpha(vz_i, \omega_i, \theta_i) \cdot \Delta t \\
\theta_i + \omega_i \cdot \Delta t
\end{bmatrix}\\
\omega_{\text{min}} &:= \frac{g}{r} \\
\omega_{\text{min}} &:= 4.044
\end{align*}
The minimum initial velocity of 10.6 m/s was determined by trial and error. The motion damps out after 4 seconds and the particle comes to rest at 360 degrees.
2.103 The forces are:

\[ N = -N \hat{e}_r, \]

\[ \mathbf{W} = mg (\cos \beta \sin \theta \hat{e}_r + \cos \beta \cos \theta \hat{e}_\theta + \sin \beta \hat{k}) \]

The equations of motion are:

\[-N + mg \cos \beta \sin \theta = -mr\ddot{\theta}^2 \]
\[mg \cos \beta \cos \theta = mr\ddot{\theta} \]
\[mg \sin \beta = m\ddot{z} \]

The data is:

\[ m = 2 \text{ kg} \quad \theta(0) = 0 \]
\[ r = 1 \text{ m} \]
\[ \beta = 20^\circ \]

The \( z \) equation can be quickly integrated:

\[ \ddot{z} = g \sin \beta \]
\[ \dot{z} = g(\sin \beta) t \]
\[ z = g(\sin \beta) t^2 / 2 \]

The MATLAB code is:

```matlab
function yprime = ds2pt103(t,y)
g = 9.81;
R = 1;
beta = 20*pi/180;
yprime(1) = y(2);
yprime(2) = (g/R)*cos(beta)*cos(y(1));
return
```
The Mathcad code is:

\[
i := 0 \ldots 2500 \\
\Delta t := 0.001 \\
t_i := i \cdot \Delta t \\
\beta := 20 \, \text{deg} \\
r := 1 \\
g := 9.81 \\
\alpha(\theta) := \frac{g}{r} \cos(\beta) \cdot \cos(\theta) \\
\begin{bmatrix}
\omega_0 \\
\theta_0 \\
\omega_i + 1 \\
\theta_i + 1
\end{bmatrix} := \\
\begin{bmatrix}
0 \\
0 \\
\omega_i + \alpha(\theta_i) \cdot \Delta t \\
\theta_i + \omega_i \cdot \Delta t
\end{bmatrix} \\
n := 0 \ldots 2500 \\
z_n := g \cdot \sin(\beta) \cdot \frac{(t_n)^2}{2}
\]

![Angular position vs time](image1)

![Position down sluiceway](image2)
2.104 We are given: \( \mu_k = 0.3 \)

Friction acts in a direction opposite to the velocity
\[
\mathbf{v} = r\omega \hat{e}_\theta + vz \hat{k}
\]
\[
\mathbf{N} = -N\hat{e}_r
\]
\[
\mathbf{W} = mg(\cos \beta \sin \theta \hat{e}_r + \cos \beta \cos \theta \hat{e}_\theta + \sin \beta \hat{k})
\]
\[
\mathbf{f} = -N\mu \frac{\mathbf{v}}{||\mathbf{v}||}
\]

The equations of motion are:
\[
-N + mg \cos \beta \sin \theta = -mr\ddot{\omega}^2
\]
\[
mg \cos \beta \cos \theta - \mu N \frac{r\omega}{\sqrt{(r\omega)^2 + vz^2}} = mr\ddot{\theta}
\]
\[
mg \sin \beta - \mu N \frac{vz}{\sqrt{(r\omega)^2 + vz^2}} = m\ddot{z}
\]

Therefore \( N = m(g \cos \beta \sin \theta + r\omega^2) \) and
\[
\alpha(vz, \omega, \theta) = \frac{1}{r} \left[ g \cos \beta \cos \theta - \frac{\mu N}{m} \frac{r\omega}{\sqrt{(r\omega)^2 + vz^2}} \right]
\]
\[
a(vz, \omega, \theta) = g \sin \beta - \frac{\mu N}{m} \frac{vz}{\sqrt{(r\omega)^2 + vz^2}}.
\]

The MATLAB code is:
```matlab
function yprime = ds2pt103(t,y)

m = 2;
g = 9.81;
R = 1;
mu = 0.3;
beta = 20*pi/180;

th = y(1);
z = y(2);
thdot = y(3);
zdot = y(4);

v = sqrt((R*thdot)^2 + zdot^2);
N = m*g*cos(beta)*sin(th)+m*R*thdot^2;

yprime(1) = thdot;
yprime(2) = zdot;
yprime(3) = (m*g*cos(beta)*cos(th)-mu*N*R*thdot/v)/m/R;
yprime(4) = (m*g*sin(beta)-mu*n*zdot/v)/m;

return
```
The Mathcad code is:

\[
\begin{align*}
i &:= 0 \ldots 6000 \\
\Delta t &:= 0.001 \\
t_i &:= i \cdot \Delta t \\
\beta &:= 20\text{-deg} \\
r &:= 1 \\
g &:= 9.81 \\
N(\omega, \theta) &:= g \cdot \cos(\beta) \cdot \sin(\theta) + r \cdot \omega^2 \\
\alpha(vz,\omega,\theta) &:= \frac{1}{r} \left[ g \cdot \cos(\beta) \cdot \cos(\theta) - \mu \cdot N(\omega,\theta) \cdot \frac{r \cdot \omega}{\sqrt{(r \cdot \omega)^2 + vz^2}} \right] \\
a(vz,\omega,\theta) &:= g \cdot \sin(\beta) - \mu \cdot N(\omega,\theta) \cdot \frac{vz}{\sqrt{(r \cdot \omega)^2 + vz^2}} \\
\begin{bmatrix}
vz_0 \\
z_0 \\
\omega_0 \\
\theta_0
\end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} vz_{i+1} \\
z_{i+1} \\
\omega_{i+1} \\
\theta_{i+1}
\end{bmatrix} &= \begin{bmatrix} vz_i + a(vz_i,\omega_i,\theta_i) \cdot \Delta t \\
z_i + vz_i \cdot \Delta t \\
\omega_i + \alpha(vz_i,\omega_i,\theta_i) \cdot \Delta t \\
\theta_i + \omega_i \cdot \Delta t
\end{bmatrix}
\end{align*}
\]
2.105 We have \( m = 2 \text{ kg} \), \( R = 1.5 \text{ m} \), \( \mathbf{v}(0) = 10\hat{\mathbf{e}}_\theta \), \( \phi(0) = 90^\circ \)

![Figure S2.105](image_url)

The forces are:

\[
\mathbf{N} = -N\hat{\mathbf{e}}_R
\]

\[
\mathbf{W} = mg(\sin \phi \hat{\mathbf{e}}_\phi - \cos \phi \hat{\mathbf{e}}_R)
\]

The coordinate accelerations are

\[
a_R = \ddot{R} - R\phi^2 - R\sin^2 \phi \dot{\theta}^2
\]

\[
a_\phi = R\ddot{\phi} + 2R\dot{\phi} - R\sin \phi \cos \phi \dot{\theta}^2
\]

\[
a_\theta = R\sin \phi \dot{\theta} + 2R\dot{\theta} \sin \phi \phi \dot{\phi} \cos \phi
\]

Therefore 

\[
-N - mg \cos \phi = -mR(\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2)
\]

\[
mg \sin \phi = mR(\ddot{\phi} - \sin \phi \cos \phi \dot{\theta}^2)
\]

\[
0 = R \sin \theta \ddot{\theta} + 2R\dot{\phi} \dot{\theta} \cos \phi
\]

From sample problem 2.22, the last equation

\[
\dot{\theta} = \frac{\sin \phi \nu_{\theta 0}}{R \sin^2 \phi} = \frac{10}{R \sin^2 \phi}
\]

\[
\ddot{\phi} = \frac{g}{R} \sin \phi + \frac{\cos \phi \nu_{\phi 0}^2 \sin^2 \phi_0}{R^2 \sin^3 \phi}
\]

The MATLAB code is:

```matlab
function yprime = ds2pt105(t,y)
g = 9.81;
R = 1.5;
ph = y(1);
th = y(2);
phdot = y(3);
yprime(1) = y(3);
yprime(2) = 10/(R*sin(ph)^2);
yprime(3) = -(g/R)*sin(ph) + 100*cos(ph)/(R^2*sin(ph)^3);
return
```

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The Mathcad code is (first define R and g in the code):

\[
\begin{align*}
\Delta t & := 0.001 \\
t_i & := i \cdot \Delta t \\
d\Phi (\phi) & := g \cdot \frac{\sin(\phi)}{R} + \frac{100 \cdot \cos(\phi)}{R^2 \cdot \sin(\phi)^3} \\
\begin{bmatrix}
\frac{d\phi_0}{dt} \\
\phi_0
\end{bmatrix} & := \begin{bmatrix} 0 \\
90 \cdot \text{deg} \end{bmatrix} \\
\begin{bmatrix}
\frac{d\phi_{i+1}}{dt} \\
\phi_{i+1}
\end{bmatrix} & := \begin{bmatrix} \frac{d\phi_i + d\Phi(\phi_i) \cdot \Delta t}{dt} \\
\phi_i + d\phi_i \cdot \Delta t \end{bmatrix} \\
d\theta (\phi) & := \frac{10}{R \cdot \sin(\phi)^2} \\
\theta_0 & := 0 \\
\theta_{i+1} & := \theta_i + d\theta(\phi_i) \cdot \Delta t \\
x_i & := R \cdot \sin(\phi_i) \cdot \cos(\theta_i) \\
y_i & := R \cdot \sin(\phi_i) \cdot \sin(\theta_i) \\
z_i & := R \cdot \cos(\phi_i) \\
\phi_0 & = 1.571
\end{align*}
\]
2.106 We are given: $R = 1.5$, $\dot{\theta}_0 = \frac{10}{15}$, $\theta_0 = 0$, $\phi_0 = 90^\circ$

\[ v = R \dot{\phi} \hat{e}_\phi + R \sin \phi \dot{\theta} \hat{e}_\theta \]

\[ N = -N \hat{e}_R \]

\[ W = mg (\sin \phi \hat{e}_\phi - \cos \phi \hat{e}_r) \]

\[ f = -\mu N \frac{\dot{v}}{|v|} \]

\[-N - mg \cos \phi = -mR(\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2) \]

\[ mg \sin \phi - \mu N \frac{R \dot{\phi}}{|v|} = mR(\ddot{\phi} - \sin \phi \cos \phi \dot{\theta}^2) \]

\[-\mu N \frac{R \sin \phi \dot{\phi}}{|v|} = mR(\sin \phi \ddot{\theta} + 2\dot{\phi} \dot{\theta} \cos \phi) \]

The mass drops out.

\[ \frac{N}{m} = R(\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2) - g \cos \phi \]

Therefore

\[ \ddot{\phi} = \frac{v}{R} \sin \phi - \mu R(\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2) \frac{\dot{\phi}}{|v|} + \sin \phi \cos \phi \dot{\theta}^2 \]

\[ \ddot{\theta} = \frac{1}{\sin \phi} \left[ -\mu \frac{N \sin \phi \dot{\phi}}{m v} - 2\dot{\phi} \dot{\theta} \cos \phi \right] \]

The MATLAB code is:

```matlab
function yprime = ds2pt105(t,y)
g = 9.81;
R = 1.5;
mu = 0.3;
ph = y(1);
th = y(2);
phdot = y(3);
thdot = y(4);
v = R*sqrt(phdot^2+sin(ph)^2*thdot^2);
N = g*cos(ph)+v^2/R;
yprime(1) = y(3);
yprime(2) = y(4);
yprime(3) = - (g/R)*sin(ph) -mu*Nphdot/v+sin(ph)*cos(ph)*thdot^2;
yprime(4) = -mu*N*thdot/v-2*phdot*thdot/tan(ph);
return
```

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The Mathcad code is:

\[
\begin{align*}
R & := 1.5 \quad g := 9.81 \quad \mu := 0.3 \\
i & := 0 \ldots 4000 \\
\Delta t & := 0.001 \\
t_i & := i \cdot \Delta t \\
N (d\phi, \phi, d\theta) & := R \cdot (d\phi^2 + \sin(\phi)^2 d\theta) - g \cdot \cos(\phi) \\
v (d\phi, \phi, d\theta) & := \sqrt{(R \cdot d\phi)^2 + (R \cdot \sin(\phi) \cdot d\theta)^2} \\
dd\phi (d\phi, \phi, d\theta) & := -\frac{g}{R} \cdot \sin(\phi) - \mu \cdot N (d\phi, \phi, d\theta) \cdot \frac{d\phi}{v (d\phi, \phi, d\theta)} + \sin(\phi) \cdot \cos(\phi) \cdot d\theta^2 \\
dd\theta (d\phi, \phi, d\theta) & := \frac{-1}{\sin(\phi)} \left( \mu \cdot N (d\phi, \phi, d\theta) \cdot \sin(\phi) \cdot \frac{d\theta}{v (d\phi, \phi, d\theta)} + 2 \cdot d\phi \cdot d\theta \cdot \cos(\phi) \right)
\end{align*}
\]

\[
\begin{bmatrix}
\phi_0 \\
\theta_0 \\
\phi_1 + 1 \\
\theta_1 + 1 \\
x_i \\
y_i \\
z_i
\end{bmatrix} := 
\begin{bmatrix}
0 \\
90 \cdot \text{deg} \\
10 \\
1.5 \\
0 \\
\phi_i + d\phi_i \cdot \Delta t \\
\theta_i + d\theta_i \cdot \Delta t \\
R \cdot \sin(\phi_i) \cdot \cos(\theta_i) \\
R \cdot \sin(\phi_i) \cdot \sin(\theta_i) \\
R \cdot \cos(\phi_i)
\end{bmatrix} \cdot \Delta t
\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2.107.png}
\caption{FIGURE S2.107}
\end{figure}

2.107 Solution:
\[ N = -N\hat{e}_\phi \]
\[ W = mg(-\cos \phi \hat{e}_e + \sin \phi \hat{e}_\phi) \]
\[ f = -\mu N \frac{\dot{y}}{\|\dot{y}\|} \]
\[ \mathbf{a}_R = \ddot{R} - R\dot{\phi}^2 - R \sin^2 \phi \dot{\theta}^2 \]
\[ \mathbf{a}_\phi = R\ddot{\phi} + 2\dot{R}\dot{\phi} - R \sin \phi \cos \phi \dot{\theta}^2 \]
\[ \mathbf{a}_\theta = R \sin \phi \ddot{\theta} + 2\dot{R}\dot{\theta} \sin \phi + 2R\phi \dot{\phi} \cos \phi \]
\[ \phi = \text{const} \]
\[ v = \dot{R}\hat{e}_R + R \sin \phi \dot{\phi} \hat{e}_\theta \]
\[-mg \cos \phi - \mu N \frac{\ddot{R}}{v} = m(\ddot{R} - R \sin^2 \phi \dot{\theta}^2) \]
\[-N + mg \sin \phi = -mR \sin \phi \cos \phi \dot{\theta}^2 \]
\[-\mu N \frac{R \sin \phi \ddot{\phi}}{v} = m(R \sin \phi \ddot{\theta} + 2\dot{R}\dot{\theta} \sin \phi) \]
\[ N/m(d\theta, R) = g \sin \phi + R \sin \phi \cos \phi \dot{\theta}^2 \]
\[ v(dR, R, d\theta) = \sqrt{dR^2 + (R \sin \phi d\theta)^2} \]
\[ \ddot{R} = -g \cos \phi + R \sin^2 \phi \dot{\theta}^2 - \mu \frac{N}{m} \frac{R}{v} \]
\[ \ddot{\theta} = -\mu \frac{v}{m} \frac{\dot{\theta}}{v} - \frac{2\dot{R}\dot{\phi}}{R} \]

The MATLAB code is:

```matlab
function yprime = ds2pt107(t,y)
g = 9.81;
mu = 0;
phi = pi/6;
r = y(1);
tha = y(2);
rddot = y(3);
tha = y(4);
v = sqrt(rddot^2+(r*sin(phi)*tha)^2);
N = g*sin(phi)+r*sin(phi)*cos(phi)*tha^2;
yprime(1) = rddot;
yprime(2) = tha;
yprime(3) = -mu*N*rddot / v - g*cos(phi)+r*sin(phi)^2*tha^2;
yprime(4) = -mu*N*tha/v - 2*rddot*tha/r;
return
```
The Mathcad code is:

\[ i := 0..6000 \]
\[ \Delta t := 0.001 \]
\[ t_i := i \cdot \Delta t \]
\[ g := 9.81 \quad \mu := 0.0 \]
\[ \phi := 30\,\text{deg} \]
\[ N(d\theta, R) := g \cdot \sin(\phi) + R \cdot \sin(\phi) \cdot \cos(\phi) \cdot d\theta^2 \]
\[ R_0 := \frac{2}{\cos(\phi)} \]
\[ v(d\theta, dR, R) := \sqrt{dR^2 + (R \cdot \sin(\phi) \cdot d\theta)^2} \]
\[ ddR(d\theta, dR, R) := -g \cdot \cos(\phi) + R \cdot \sin(\phi)^2 \cdot d\theta - \mu \cdot N(d\theta, R) \cdot \frac{dR}{v(d\theta, dR, R)} \]
\[ dd\theta(d\theta, dR, R) := \mu \cdot N(d\theta, R) \cdot \frac{d\theta}{v(d\theta, dR, R)} - 2 \cdot \frac{dR \cdot d\theta}{R} \]

\[
\begin{bmatrix}
    dR_0 \\
    R_0 \\
    d\theta_0 \\
    \theta_0
\end{bmatrix} :=
\begin{bmatrix}
    0 \\
    R_0 \\
    10 \\
    R_0 \cdot \sin(\phi) \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    dR_{i+1} \\
    R_{i+1} \\
    d\theta_{i+1} \\
    \theta_{i+1}
\end{bmatrix} :=
\begin{bmatrix}
    dR_i + ddR(d\theta_i, dR_i, R_i) \cdot \Delta t \\
    R_i + dR_i \cdot \Delta t \\
    d\theta_i + dd\theta(d\theta_i, dR_i, R_i) \cdot \Delta t \\
    \theta_i + d\theta_i \cdot \Delta t
\end{bmatrix}
\]

\[ x_i := R_i \cdot \sin(\phi) \cdot \cos(\theta_i) \]
\[ y_i := R_i \cdot \sin(\phi) \cdot \sin(\theta_i) \]
\[ z_i := R_i \cdot \cos(\theta_i) \]

Path of Motion

\[ x, y, z \]
2.108 Use the analysis and codes of 2.107 with a new friction coefficient. The codes are essentially the same, so only the Mathcad plot is shown.

![Path of Motion](image)

2.109 Solution:

\[ \mathbf{N} = N\hat{e}_R \]
\[ \mathbf{W} = mg(-\cos \phi \hat{e}_R + \sin \phi \hat{e}_\phi) \]
\[ \mathbf{V} = R\dot{\phi}\hat{e}_\phi + R\sin \phi \hat{e}_\theta \]
\[ \mathbf{f} = -\mu N\frac{\dot{R}}{|\dot{R}|} \]

\[ N - mg \cos \phi = -m(R\ddot{\phi}^2 + R\sin^2 \phi \dot{\theta}^2) \]
\[ mg \sin \phi - \mu N R\dot{\phi} \frac{R}{v} = m(R\ddot{\phi} - R \sin \phi \cos \phi \dot{\theta}^2) \]
\[ -\mu N R\sin \phi \frac{R\dot{\phi}}{v} = m(R \sin \phi \ddot{\theta} + 2R\dot{\phi} \dot{\theta} \cos \phi) \]
\[ v(d\phi, \phi, d\theta) = R \cdot \sqrt{d\phi^2 + (\sin(\phi)d\theta)^2} \]
\[ N/m = g \cos \phi - R(\dot{\phi}^2 + \sin^2 \phi \theta^2) \]
\[ \ddot{\phi} = \frac{g}{R} \sin \phi - \mu \frac{N}{m} \frac{\dot{\phi}}{v} + \sin \phi \cos \theta \frac{\sin^2 \phi}{\sin \phi} \]
\[ \ddot{\theta} = -\mu \frac{N}{m} \frac{\dot{\theta}}{v} - 2 \dot{\phi} \dot{\theta} \frac{\cos \phi}{\sin \phi} \]

The MATLAB code is:

```matlab
function yprime = ds2pt107(t,y)
    g = 9.81;
    mu = 0;
    r = 6;
    ph = y(1);
    th = y(2);
    phdot = y(3);
    thdot = y(4);
    v = r*sqrt(phdot^2+(sin(ph)*thdot)^2);
    N = g*cos(ph) -v^2/r;
    yprime(1) = phdot;
    yprime(2) = thdot;
    yprime(3) = -mu*N*phdot/v + (g/4)*sin(ph) +sin(ph)*cos(ph)*thdot^2;
    yprime(4) = -mu*N*thdot/v -2*phdot*thdot/tan(ph);
    return
```
The Mathcad code is:

\[
i := 0..900 \\
\Delta t := 0.001 \\
t_i := i \cdot \Delta t \\
R := 6 \\
g := 9.81 \\
\mu := 0.0 \\

v(d\phi, \phi, d\theta) := R \cdot \sqrt{d\phi^2 + (\sin(\phi) \cdot d\theta)^2} \\
N(d\phi, \phi, d\theta) := g \cdot \cos(\phi) - R \cdot [d\phi^2 + (\sin(\phi) \cdot d\theta)^2] \\
dd\phi(d\phi, \phi, d\theta) := \frac{g}{R} \cdot \sin(\phi) - \mu \cdot (N(d\phi, \phi, d\theta)) \cdot \frac{d\theta}{v(d\phi, \phi, d\theta)} + \sin(\phi) \cdot \cos(\phi) \cdot d\theta^2 \\
dd\theta(d\phi, \phi, d\theta) := -\mu \cdot N(d\phi, \phi, d\theta) \cdot \frac{d\theta}{v(d\phi, \phi, d\theta)} - 2 \cdot d\phi \cdot d\theta \cdot \frac{\cos(\phi)}{\sin(\phi)}
\]

The particle will fall off the surface when the normal force goes to zero and the position coordinates are shown.

\[
\begin{bmatrix}
d\phi_0 \\
\phi_0 \\
d\theta_0 \\
\theta_0
\end{bmatrix}
:=
\begin{bmatrix}
-0.5 \\
10 \cdot \text{deg} \\
4 \\
6 \cdot \sin(10 \cdot \text{deg})
\end{bmatrix}
\]

\[
\begin{bmatrix}
d\phi_{i+1} \\
\phi_{i+1} \\
d\theta_{i+1} \\
\theta_{i+1}
\end{bmatrix}
:=
\begin{bmatrix}
d\phi_i + dd\phi(d\phi_i, \phi_i, \theta_i) \cdot \Delta t \\
\phi_i + d\phi_i \Delta t \\
d\theta_i + dd\theta(d\phi_i, \phi_i, \theta_i) \cdot \Delta t \\
\theta_i + d\theta_i \Delta t
\end{bmatrix}
\]

\[
N(d\phi_i, \phi_i, d\theta_i) = -3.95 \cdot 10^{-3}
\]

\[
\frac{\phi_{866}}{\text{deg}} = 41.541 \\
\frac{\theta_{866}}{\text{deg}} = 75.735
\]

The particle will fall off the surface when the normal force goes to zero and the position coordinates are shown.
2.110 Use the same codes and analysis as in 2.109. The MATLAB code becomes:

```matlab
clear all
format short 3
format compact
g = 9.81;
r = 6;
ph0 = pi/18;
th0 = 0;
phdot0 = -0.5/r;
thdot0 = 4/r/sin(ph0);
[t,Y] = ode45('ds2pt109',0,1,[ph0 ; th0; phdot0; thdot0]);
ph = Y(:,1);
th = Y(:,2);
phdot = Y(:,3);
thdot = Y(:,4);
v = r*sqrt(phdot.^2+(sin(ph).*thdot).^2);
N = g*cos(ph( - v.^2/r;
figure(1),plot(t,N)
```
The Mathcad code is:

```
i := 0..2000
Dt := 0.001
t := i * Dt
R := 6
G := 9.81
µ := 0.2
v(δφ, φ, δθ) := R * √(δφ^2 + (sin(φ) * δθ)^2)
N(δφ, φ, δθ) := G * cos(φ) - R * [δφ^2 + (sin(φ) * δθ)^2]
ddθ(δφ, φ, δθ) := \(\frac{g}{R} \cdot \sin(φ) - µ \cdot (N(δφ, φ, δθ)) \cdot \frac{dθ}{v(δφ, φ, δθ)} + \sin(φ) \cdot \cos(φ) \cdot δθ^2\)
ddθ(δφ, φ, δθ) := -µ * N(δφ, φ, δθ) * \(\frac{dθ}{v(δφ, φ, δθ)} - 2 \cdot δφ \cdot \cos(φ) \cdot δθ^2\)

\[
\begin{bmatrix}
δφ_0 \\
δθ_0 \\
θ_0
\end{bmatrix}
\begin{bmatrix}
-0.5 & 6 \\
10-deg & 4 \\
6-\sin(10-deg) & 0
\end{bmatrix}
\begin{bmatrix}
δφ_i + 1 \\
δθ_i + 1 \\
θ_i + 1
\end{bmatrix}
\begin{bmatrix}
\deltaφ_i + dd\phi(\deltaφ_i, \phi_i, \deltaθ_i) \cdot Δt \\
\deltaθ_i + ddθ(\deltaφ_i, \phi_i, \deltaθ_i) \cdot Δt \\
\theta_i + dθi \cdot Δt
\end{bmatrix}
```

The mass stays on the surface for a greater length of time when friction is present. If the friction is too great, the particle will not leave the surface before it stops.
The data is:

\[ \phi(0) = 30^\circ \quad \theta(0) = 0 \quad m = 3 \text{ kg} \quad R(0) = 2 \quad \nu_\theta = 0.5, \text{ therefore } \dot{\theta}(0) = 0.5. \]

\[ F_s = -k(R - R_0) \hat{e}_R \]

\[ W = mg(\cos \phi \hat{e}_R - \sin \phi \hat{e}_\phi) \]

The equation of motion become:

\[ m[\ddot{R} - R \dot{\phi}^2 - R \sin^2 \phi \dot{\theta}^2] = mg \cos \phi - k(R - 2) \]

\[ m[R \ddot{\phi} + 2R \dot{\phi} - \sin \phi \cos \phi \dot{\theta}^2] = -mg \sin \phi \]

\[ m[R \sin \phi \ddot{\theta} + 2R \dot{\theta} \sin \phi + 2R \dot{\phi} \dot{\theta} \cos \phi] = 0, \text{ or:} \]

\[
\begin{bmatrix}
\ddot{R} \\
\ddot{\phi} \\
\ddot{\theta}
\end{bmatrix}
= -g \begin{bmatrix}
\cos \phi \\
\frac{\sin \phi}{R} - \frac{\hat{R}}{R} \\
\frac{\sin \phi \cos \phi}{R^2}
\end{bmatrix} \hat{e}_R
\]

The last equation can be written

\[ \frac{1}{R \sin \phi} \frac{d}{dt}[R^2 \sin^2 \phi \dot{\theta}] = 0 \]

Therefore

\[ R^2 = \sin^2 \phi \dot{\theta} = h \]

\[ h = R_0^2 \sin^2 \phi \dot{\theta}_0 = R_0 \sin \phi_0 \nu_{\theta 0} \]

For this case \( h = 2 \sin 30^\circ(0.5) = 0.5 \)

\[ \dot{\theta} = \frac{h}{R^2 \sin^2 \phi} \]

The differential equations may be written as

\[ \ddot{R} = g \cos \phi - \frac{k}{m}(R - 2) + R \ddot{\phi}^2 + \frac{R \sin^2 \phi \dot{\theta}^2}{R^2 \sin^2 \phi} \]

\[ \ddot{R} = g \cos \phi - \frac{k}{m}(R - 2) + R \ddot{\phi}^2 + \frac{h^2}{R^3 \sin^2 \phi} \]

\[ \ddot{\phi} = -\frac{g}{R} \sin \phi - \frac{2R \dot{\phi}}{R^3} + \frac{\sin \phi \cos \phi \dot{\theta}^2}{R^4 \sin^2 \phi} \]
The MATLAB code is:

```matlab
function yprime = ds2pt111(t,y)
    m = 3;
k = 200;
g = 9.81;
R = 2;
v0 = 0.5;
h = v0*R*sin(pi/6);
    r = y(1);
    ph = y(2);
    th = y(3);
    rdot = y(4);
    phdot = y(5);
    thdot = h/r*sin(ph)^2;
    yprime(1) = rdot;
    yprime(2) = phdot;
    yprime(3) = thdot;
    yprime(4) = -(k/m)*(r-R)+g*cos(ph) + r*phdot + 4*sin(ph)^2*thdot^2;
    yprime(5) = -(g/r)*sin(ph) -2*(rdot/r)*phdot + sin(ph)*cos(ph)*thdot^2;
    return
```
The Mathcad code is:

\[
\begin{align*}
    h & := 0.5 \quad g := 9.81 \quad k := 200 \quad m := 3 \\
i & := 0 \ldots 5000 \\
\Delta t & := 0.001 \\
t_i & := i \cdot \Delta t \\
d\mathbf{R}(dR, R, d\phi, \phi) & := g \cdot \cos(\phi) - \frac{k}{m} (R - 2) + R \cdot d\phi^2 + \frac{h^2}{R^3 \cdot \sin(\phi)^2} \\
d\phi (dR, R, d\phi, \phi) & := -\frac{g}{R} \sin(\phi) - 2 \cdot \frac{dR \cdot d\phi}{R} + \frac{\cos(\phi) \cdot h^2}{R^4 \cdot \sin(\phi)^3} \\
\begin{bmatrix}
    dR_0 \\
    R_0 \\
    d\phi_0 \\
    \phi_0
\end{bmatrix} & := \begin{bmatrix}
    0 \\
    2 \\
    0 \\
    30\text{-deg}
\end{bmatrix} \\
\begin{bmatrix}
    dR_{i+1} \\
    R_{i+1} \\
    d\phi_{i+1} \\
    \phi_{i+1}
\end{bmatrix} & := \begin{bmatrix}
    dR_i + ddR(dR_i, R_i, d\phi_i, \phi_i) \cdot \Delta t \\
    R_i + dR_i \cdot \Delta t \\
    d\phi_i + d\phi(dR_i, R_i, d\phi_i, \phi_i) \cdot \Delta t \\
    \phi_i + d\phi_i \cdot \Delta t
\end{bmatrix} \\
i & := 0 \ldots 5000 \\
\theta_0 & := 0 \\
\theta_{i+1} & := \frac{h}{(R_i)^2 \cdot \sin(\phi_i)^2} \cdot \Delta t + \theta_i \\
x_i & := R_i \cdot \sin(\phi_i) \cdot \cos(\theta_i) \\
y_i & := R_i \cdot \sin(\phi_i) \cdot \sin(\theta_i) \\
z_i & := R_i \cdot \cos(\phi_i)
\end{align*}
\]
Path of motion

\[ y_i, x_i, -z_i \]
The MATLAB code is:

```matlab
clear all
format short e
format compact
N = 100;
W = 3000;
g = 32.2;
s = linspace(0,6000,N);
v = 88;

th = s/2000 +exp(-s/1000);
be = (pi/12)*sin(pi*s/3000);

th = 1/2000 -(1/100)*exp(-s/1000);
be = (pi^2/36000)*cos(pi*s/3000);

Gvec = [ -sin(th).*cos(be).*th.*cos(th).*sin(be.*be);
          cos(th).*cos(be).*th.*sin(th).*sin(be.*be);... 
          cos(be).*be];

for i = 1:N,
    mag_Fn(i) = (W/g)*v^2*norm(Gvec(:,i));
end

Figure (1),plot(s,mag_Fn)
xlabel('distance along road (ft)')
ylabel('normal force (lbf)')
grid
```
The Mathcad code is:

```mathcad
s := 0, 10..6000
W := 6000
g := 32.2

θ(s) := \frac{s}{2000} + e^{1000}

β(s) := \frac{\pi}{12} \sin\left(\frac{\pi \cdot s}{3000}\right)

\frac{dθ(s)}{ds} := \frac{1}{2000} - \frac{1}{1000} e^{1000}

\frac{dβ(s)}{ds} := \frac{\pi}{36000} \cos\left(\frac{\pi \cdot s}{3000}\right)

Γ(s) := \left[ \begin{array}{c}
-\sin(θ(s)) \cdot \cos(β(s)) \cdot dθ(s) - \cos(θ(s)) \cdot \sin(β(s)) \cdot dβ(s) \\
\cos(θ(s)) \cdot \cos(β(s)) \cdot dθ(s) - \sin(θ(s)) \cdot \sin(β(s)) \cdot dβ(s)
\end{array} \right]

v(s) := 88

F_n(s) := \frac{W}{g} \cdot v(s)^2 \cdot Γ(s)
```

![Graph](image-url)
2.113 Solution:

\[ a = 2, \quad 0 \leq s \leq 3000 \]

\[ v = \sqrt{2as} \]

\[ v(3000) = 109.5 \text{ ft/s (75 mph)} \]

\[ a = -1, \quad 3001 \leq s \leq 6000 \]

\[ \int_{109.5}^{v} v \, dv = \int_{s}^{3000} a \, ds \]

\[ \frac{v^2}{2} - \frac{(109.5)^2}{2} = -s + 3000 \]

\[ v = \sqrt{109.5^2 - 2(s - 3000)} \]

The Mathcad code follows:

\[
\begin{align*}
\text{s} & := 0, 10 \ldots 6000 \\
W & := 6000 \\
g & := 32.2 \\
\theta(s) & := \frac{s}{2000} + e^{1000} \\
\beta(s) & := \frac{\pi}{12} \sin\left(\frac{\pi \cdot s}{3000}\right) \\
d\theta(s) & := \frac{1}{2000} - \frac{1}{1000} e^{1000} \\
d\beta(s) & := \frac{\pi}{36000} \cos\left(\frac{\pi \cdot s}{3000}\right) \\
\Gamma(s) & := \begin{bmatrix}
-s \sin(\theta(s)) \cdot \cos(\beta(s)) \cdot d\theta(s) - \cos(\theta(s)) \cdot \sin(\beta(s)) \cdot d\beta(s) \\
\cos(\theta(s)) \cdot \cos(\beta(s)) \cdot d\theta(s) - \sin(\theta(s)) \cdot \sin(\beta(s)) \cdot d\beta(s) \\
\cos(\beta(s)) \cdot d\beta(s)
\end{bmatrix} \\
v(s) & := \Phi(3000 - s) \cdot \sqrt{4s + \Phi(s - 3001) \cdot \sqrt{109.5^2 - 2(s - 3000)}} \\
F_n(s) & := \frac{W}{g} v(s)^2 \cdot \Gamma(s)
\end{align*}
\]
Using the analysis of 2.113 the MATLAB code is:

```matlab
clear all
format short e
format compact
N = 100;
W = 3000;
g = 32.2;
s = linspace(0,6000,N);
v = sqrt( 4*s.*(s>=0) -6*(s-3000).*((s>=3000));
a = 2*(s>=0) -3*(s>=3000);
t = s/2000 +exp(-s/1000);
b = (pi/12)*sin(pi*s/3000);
t_ = 1/2000 - (1/1000)*exp(-s/1000);
b_ = (pi^2/36000)*cos(pi*s/3000);
avec = [...
a.*cos(t).*cos(b)+v.^2.*(-sin(t).*cos(b).*t.*cos(t).*sin(b).*b_);
a.*sin(t).*cos(b)+v.^2.*( cos(t).*cos(b).*t.*sin(t).*sin(b).*b_);
a.*sin(b)+v.^2.*cos(b).*b_;
];
mag_a = sqrt(sum((avec.*avec)'));
figure(1),plot(s,mag_a)
xlabel('distance along road (ft)')
ylabel('acceleration (ft/s^2)')
grid
```
The Mathcad code is:

\[ s := 0, 10 \ldots 6000 \]
\[ W := 6000 \]
\[ g := 32.2 \]
\[ \theta (s) := \frac{s}{2000} + e^{1000} \]
\[ \beta (s) := \frac{\pi}{12} \sin \left( \frac{\pi \cdot s}{3000} \right) \]
\[ d\theta (s) := \frac{1}{2000} - \frac{1}{1000} e^{1000} \]
\[ d\beta (s) := \frac{\pi}{36000} \cos \left( \frac{\pi \cdot s}{3000} \right) \]
\[ \Gamma (s) := \begin{bmatrix}
-\sin (\theta (s)) \cdot \cos (\beta (s)) \cdot d\theta (s) - \cos (\theta (s)) \cdot \sin (\beta (s)) \cdot d\beta (s) \\
\cos (\theta (s)) \cdot \cos (\beta (s)) \cdot d\theta (s) - \sin (\theta (s)) \cdot \sin (\beta (s)) \cdot d\beta (s)
\end{bmatrix} \cdot \cos (\beta (s)) \cdot d\beta (s) \]
\[ v (s) := \Phi (3000 - s) \cdot \sqrt{4 \cdot s + \Phi (s - 3001) \cdot \sqrt{109.5^2 - 2 \cdot (s - 3000)}} \]
\[ a_n (s) := 1 \cdot v (s)^2 \cdot \Gamma (s) \]
\[ a_t (s) := \Phi (3000 - s) \cdot 2 - \Phi (s - 3001) \]
\[ a (s) := \sqrt{(a_n (s))^2 + a_t (s)^2} \]

![Acceleration Magnitude vs position](image-url)
2.115 Solution:

\[ F = -mg\hat{k} + N\hat{n} - \mu \hat{t} = ma \]
\[ \hat{n} = \frac{r}{|r|} \quad f = -\mu N \frac{v}{|v|} \]
\[ \hat{t} = \cos \theta(s) \cos \beta(s) \hat{i} + \sin \theta(s) \cos \beta(s) \hat{j} + \sin \beta(s) \hat{k} \]
\[ N = mg\hat{k} \cdot \hat{n} + m \cdot \frac{v^2}{p} \]
\[ a(s) = \frac{-mg\hat{k} - \mu N}{m} \]

The MATLAB code is:

```matlab
function yprime = ds2pt115(t,y)
    m = 30;
g = 9.81;
mu = 0.0;
s = y(1);
v = y(2);

th = pi*s/20;
be = -(pi/3)*(1-(s/10).^2);

th_ = pi/20;
be_ = 2*pi*s/300;

kvec = [ 0 ; 0 ; 1 ];
tvec = [...
    cos(th).*cos(be);...
    sin(th).*cos(be);...
    sin(be) ];
Gvec = [...
    -sin(th).*cos(be).*th_-cos(th).*sin(be).*be_;...
    cos(th).*cos(be).*th_-sin(th).*sin(be).*be_;...
    cos(be).*be_ ];
```

FIGURE S2.115
nvec = Gvec/norm(Gvec);
N = m*g*(kvec'*nvec) + v^2*norm(Gvec);
yprime(1) = y(2);
yprime(2) = -g*(kvec'*tvec) - mu*N/m;
return

The Mathcad code is:

\[
\begin{align*}
\mathbf{m} & := 30 \\
\mathbf{\mu} & := 0 \\
\mathbf{g} & := 9.81 \\
\mathbf{d}\theta(s) & := \frac{\pi}{20} \\
\mathbf{\theta}(s) & := \frac{\pi \cdot s}{20} \\
\mathbf{d}\beta(s) & := \frac{\pi}{30} \cdot \frac{2 \cdot s}{10} \\
\mathbf{\beta}(s) & := \frac{-\pi}{3} \cdot \left[ 1 - \left( \frac{s}{10} \right)^2 \right] \\
\mathbf{\Gamma}(s) & := \begin{bmatrix}
-\sin(\mathbf{\theta}(s)) \cdot \cos(\mathbf{\beta}(s)) \cdot \mathbf{d}\theta(s) - \cos(\mathbf{\theta}(s)) \cdot \sin(\mathbf{\beta}(s)) \cdot \mathbf{d}\beta(s) \\
\cos(\mathbf{\theta}(s)) \cdot \cos(\mathbf{\beta}(s)) \cdot \mathbf{d}\theta(s) - \sin(\mathbf{\theta}(s)) \cdot \sin(\mathbf{\beta}(s)) \cdot \mathbf{d}\beta(s)
\end{bmatrix} \\
\mathbf{t}(s) & := \begin{bmatrix}
\cos(\mathbf{\theta}(s)) \cdot \cos(\mathbf{\beta}(s)) \\
\sin(\mathbf{\theta}(s)) \cdot \cos(\mathbf{\beta}(s)) \\
\sin(\mathbf{\beta}(s))
\end{bmatrix} \\
\mathbf{n}(s) & := \frac{\mathbf{\Gamma}(s)}{|\mathbf{\Gamma}(s)|} \\
\mathbf{k} & := \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \\
\mathbf{N}(\mathbf{v}, s) & := \mathbf{m} \cdot \mathbf{g} \cdot \mathbf{k} \cdot \mathbf{n}(s) + \mathbf{m} \cdot \mathbf{v} \cdot |\mathbf{\Gamma}(s)| \\
\mathbf{a}(\mathbf{v}, s) & := -\mathbf{g} \cdot \mathbf{k} \cdot \mathbf{t}(s) - \frac{\mathbf{\mu}}{\mathbf{m}} \cdot \mathbf{N}(\mathbf{v}, s) \cdot \frac{\mathbf{v}}{|\mathbf{v}|}
\end{align*}
\]
\[ i := 0 \ldots 1580 \]
\[ \Delta t := 0.001 \]
\[ t_i := i \cdot \Delta t \]

\[
\begin{bmatrix}
    v_0 \\
    s_0
\end{bmatrix}
:=
\begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    v_{i+1} \\
    s_{i+1}
\end{bmatrix}
:=
\begin{bmatrix}
    v_i + a(v_i, s_i) \cdot \Delta t \\
    s_i + v_i \cdot \Delta t
\end{bmatrix}
\]

\[ s_{1580} = 10.008 \]

\[ v_{1580} = 10.987 \text{ m/s} \]

This is the velocity at the bottom of the slide. Note the time at the bottom was found by trial and error and for this case was 1.58 s.
2.116 Use the analysis of 2.115. The MATLAB code is:

```matlab
function yprime = ds2pt116(t,y)
m = 30;
g = 9.81;
mu = 0.2;
s = y(1);
v = y(2);

th = pi*s/20;
be = -(pi/3)*(1-(s/10).^2);

th_ = pi/20;
be_ = 2*pi*s/300;

kvec = [ 0 ; 0 ; 1 ];
tvec = [...
cos(th).*cos(be);...
sin(t).*cos(be);...
sin(be) ];

Gvec = [...
-sin(th).*cos(be).*th_-cos(th).*sin(be).*be_;...
cos(th).*cos(be).*th_-sin(th).*sin(be).*be_;...
cos(be).*be_];
nvec = Gvec/norm(Gvec);
N = m*g*(kvec'*nvec) + v^2*norm(Gvec);

yprime(1) = y(2);
yprime(2) = -g*(kvec'*tvec) - mu*N/m;
return
```
The Mathcad code is:

\[
\begin{align*}
m & := 30 \\
\mu & := 0.2 \\
g & := 9.81 \\
d\theta (s) & := \frac{\pi}{20} \\
\theta (s) & := \frac{\pi \cdot s}{20} \\
d\beta (s) & := \frac{\pi}{30} \cdot 2 \cdot \frac{s}{10} \\
\beta (s) & := \frac{-\pi}{3} \left[ 1 - \left( \frac{s}{10} \right)^2 \right] \\
\Gamma (s) & := \begin{bmatrix}
-\sin (\theta (s)) \cdot \cos (\beta (s)) \cdot d\theta (s) - \cos (\theta (s)) \cdot \sin (\beta (s)) \cdot d\beta (s) \\
\cos (\theta (s)) \cdot \cos (\beta (s)) \cdot d\theta (s) - \sin (\theta (s)) \cdot \sin (\beta (s)) \cdot d\beta (s)
\end{bmatrix} \\
t(s) & := \begin{bmatrix}
\cos (\theta (s)) \cdot \cos (\beta (s)) \\
\sin (\theta (s)) \cdot \cos (\beta (s)) \\
\sin (\beta (s))
\end{bmatrix} \\
n(s) & := \frac{\Gamma (s)}{|\Gamma (s)|} \\
k & := \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \\
N(v, s) & := m \cdot g \cdot k \cdot n(s) + m \cdot v \cdot |\Gamma (s)| \\
a(v, s) & := -g \cdot k \cdot t(s) - \frac{\mu \cdot N(v, s) \cdot v}{|v|}
\end{align*}
\]
\[
\begin{align*}
i & := 0..1712 \\
\Delta t & := 0.001 \\
t_i & := i \cdot \Delta t \\
\begin{bmatrix}
v_0 \\
s_0 
\end{bmatrix} & := \begin{bmatrix} 0 \\
0 \end{bmatrix} \\
\begin{bmatrix}
v_i + 1 \\
s_i + 1 
\end{bmatrix} & := \begin{bmatrix} v_i + a(v_i, s_i) \cdot \Delta t \\
s_i + v_i \cdot \Delta t \end{bmatrix} \\
s_{1712} & = 10.001 \\
v_{1712} & = 7.966
\end{align*}
\]