Torsion

Torsional Deformations

Problem 3.2-1  A copper rod of length \( L = 18.0 \) in. is to be twisted by torques \( T \) (see figure) until the angle of rotation between the ends of the rod is \( 3.0^\circ \).

If the allowable shear strain in the copper is 0.0006 rad, what is the maximum permissible diameter of the rod?

Solution 3.2-1  Copper rod in torsion

\[ L = 18.0 \text{ in.} \]
\[ \phi = 3.0^\circ = (3.0)\left(\frac{\pi}{180}\right) \text{ rad} \]
\[ = 0.05236 \text{ rad} \]
\[ \gamma_{\text{allow}} = 0.0006 \text{ rad} \]

Find \( d_{\text{max}} \)

From Eq. (3-3):

\[ \gamma_{\text{max}} = \frac{rd\phi}{L} = \frac{d\phi}{2L} \]

\[ d_{\text{max}} = \frac{2L\gamma_{\text{allow}}}{\phi} = \frac{(2)(18.0 \text{ in.})(0.0006 \text{ rad})}{0.05236 \text{ rad}} \]

\[ d_{\text{max}} = 0.413 \text{ in.} \]

Problem 3.2-2  A plastic bar of diameter \( d = 56 \) mm is to be twisted by torques \( T \) (see figure) until the angle of rotation between the ends of the bar is \( 4.0^\circ \).

If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?
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Solution 3.2-2

Numerical data
\[ d = 56 \text{ mm} \]
\[ \gamma_a = 0.012 \text{ radians} \]
\[ \phi = 4 \left( \frac{\pi}{180} \right) \text{ radians} \]

Solution based on Equ. (3-3):
\[ L_{\text{min}} = \frac{d \phi}{2 \gamma_a} \]
\[ L_{\text{min}} = 162.9 \text{ mm} \leftarrow \]

Problem 3.2-3  A circular aluminum tube subjected to pure torsion by torques \( T \) (see figure) has an outer radius \( r_2 \) equal to 1.5 times the inner radius \( r_1 \).

(a) If the maximum shear strain in the tube is measured as \( 400 \times 10^{-6} \) rad, what is the shear strain \( \gamma_1 \) at the inner surface?

(b) If the maximum allowable rate of twist is 0.125 degrees per foot and the maximum shear strain is to be kept at \( 400 \times 10^{-6} \) rad by adjusting the torque \( T \), what is the minimum required outer radius \( (r_2)_{\text{min}} \)?

Solution 3.2-3

Numerical data
\[ r_2 = 1.5 r_1 \quad \gamma_{\text{max}} = 400 \times (10^{-6}) \text{ radians} \]
\[ \theta = 0.125 \left( \frac{\pi}{180} \right) \left( \frac{1}{12} \right) \]
\[ \theta = 1.818 \times 10^{-4} \text{ rad/m.} \]

(a) Shear strain at inner surface at radius \( r_1 \)
\[ \gamma_1 = \frac{r_1}{r_2} \gamma_{\text{max}} \]
\[ \gamma_1 = \frac{1}{1.5} \gamma_{\text{max}} \]
\[ \gamma_1 = 267 \times 10^{-6} \text{ radians} \leftarrow \]

(b) Min. required outer radius
\[ r_{2\text{min}} = \frac{\gamma_{\text{max}}}{\theta} \]
\[ r_{2\text{min}} = \frac{\gamma_{\text{max}}}{\theta} \]
\[ r_{2\text{min}} = 2.2 \text{ inches} \leftarrow \]

Problem 3.2-4  A circular steel tube of length \( L = 1.0 \text{ m} \) is loaded in torsion by torques \( T \) (see figure).

(a) If the inner radius of the tube is \( r_1 = 45 \text{ mm} \) and the measured angle of twist between the ends is 0.5°, what is the shear strain \( \gamma_1 \) (in radians) at the inner surface?

(b) If the maximum allowable shear strain is 0.0004 rad and the angle of twist is to be kept at 0.45° by adjusting the torque \( T \), what is the maximum permissible outer radius \( (r_2)_{\text{max}} \)?
**Solution 3.2-4**

**Numerical data**

$L = 1000$ mm  
$r_1 = 45$ mm  
$\phi = 0.5 \left( \frac{\pi}{180} \right)$ radians

(a) **Shear strain at inner surface**

$\gamma_1 = r_1 \frac{\phi}{L} \quad \gamma_1 = 393 \times 10^{-6}$ radians

(b) **Max. permissible outer radius**

$\phi = 0.45 \left( \frac{\pi}{180} \right)$ radians  
$\gamma_{\text{max}} = r_2 \frac{\phi}{L}$

$\gamma_{\text{max}} = 0.0004$ radians  
$r_{2\text{max}} = \gamma_{\text{max}} \frac{L}{\phi}$

$r_{2\text{max}} = 50.9$ mm

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**Problem 3.2-5**  Solve the preceding problem if the length $L = 56$ in., the inner radius $r_1 = 1.25$ in., the angle of twist is $0.5^\circ$, and the allowable shear strain is $0.0004$ rad.

**Solution 3.2-5**

**Numerical data**

$L = 56$ inches  
$r_1 = 1.25$ inches

$\phi = 0.5 \left( \frac{\pi}{180} \right)$ radians

$\gamma_a = 0.0004$ radians

(a) **Shear strain $\gamma_1$ (in radians) at the inner surface**

$\gamma_1 = r_1 \frac{\phi}{L} \quad \gamma_1 = 195 \times 10^{-6}$ radians

(b) **Maximum permissible outer radius ($r_2_{\text{max}}$)**

$\phi = 0.5 \left( \frac{\pi}{180} \right)$ radians

$\gamma_{\text{max}} = r_2 \frac{\phi}{L}$

$\gamma_a = 0.0004$ radians  
$r_{2\text{max}} = \gamma_a \frac{L}{\phi}$

$r_{2\text{max}} = 2.57$ inches
Circular Bars and Tubes

Problem 3.3-1  A prospector uses a hand-powered winch (see figure) to raise a bucket of ore in his mine shaft. The axle of the winch is a steel rod of diameter $d = 0.625$ in. Also, the distance from the center of the axle to the center of the lifting rope is $b = 4.0$ in. If the weight of the loaded bucket is $W = 100$ lb, what is the maximum shear stress in the axle due to torsion?

Solution 3.3-1  Hand-powered winch

$d = 0.625$ in.
$b = 4.0$ in.
$W = 100$ lb

Torque $T$ applied to the axle:
$T = Wb = 400$ lb-in.

Maximum shear stress in the axle

From Eq. (3-12):

$$\tau_{\text{max}} = \frac{16T}{\pi d^3}$$

$$\tau_{\text{max}} = \frac{(16)(400 \text{ lb-in.})}{\pi(0.625 \text{ in.})^3}$$

$$\tau_{\text{max}} = 8,340 \text{ psi}$$

Problem 3.3-2  When drilling a hole in a table leg, a furniture maker uses a hand-operated drill (see figure) with a bit of diameter $d = 4.0$ mm.

(a) If the resisting torque supplied by the table leg is equal to 0.3 N·m, what is the maximum shear stress in the drill bit?

(b) If the shear modulus of elasticity of the steel is $G = 75$ GPa, what is the rate of twist of the drill bit (degrees per meter)?
Solution 3.3-2 Torsion of a drill bit

\[ d = 4.0 \text{ mm} \quad T = 0.3 \text{ N} \cdot \text{m} \quad G = 75 \text{ GPa} \]

(a) Maximum shear stress

From Eq. (3-12):

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} \]

\[ \tau_{\text{max}} = \frac{16(0.3 \text{ N} \cdot \text{m})}{\pi (4.0 \text{ mm})^3} \]

\[ \tau_{\text{max}} = 23.8 \text{ MPa} \]

(b) Rate of twist

From Eq. (3-14):

\[ \theta = \frac{T}{G I_p} \]

\[ \theta = \frac{0.3 \text{ N} \cdot \text{m}}{(75 \text{ GPa}) \left( \frac{\pi}{32} \right) (4.0 \text{ mm})^4} \]

\[ \theta = 0.1592 \text{ rad/m} = 9.12^\circ /\text{m} \]

Problem 3.3-3 While removing a wheel to change a tire, a driver applies forces \( P = 25 \text{ lb} \) at the ends of two of the arms of a lug wrench (see figure). The wrench is made of steel with shear modulus of elasticity \( G = 11.4 \times 10^6 \text{ psi} \). Each arm of the wrench is 9.0 in. long and has a solid circular cross section of diameter \( d = 0.5 \text{ in.} \).

(a) Determine the maximum shear stress in the arm that is turning the lug nut (arm A).

(b) Determine the angle of twist (in degrees) of this same arm.
Solution 3.3-3  Lug wrench

- **P = 25 lb**
- **L = 9.0 in.**
- **d = 0.5 in.**
- **G = 11.4 \times 10^6 \text{ psi}**
- **T =** torque acting on arm A
- **T = P(2L) = 2(25 \text{ lb}) (9.0 \text{ in.}) = 450 \text{ lb-in.}**

(a) **Maximum shear stress**

From Eq. (3-12):

\[
\tau_{\text{max}} = \frac{16T}{\pi d^3} = \frac{(16)(450 \text{ lb-in.})}{\pi(0.5 \text{ in.})^3}
\]

\[
\tau_{\text{max}} = 18,300 \text{ psi} \quad \leftarrow
\]

(b) **Angle of twist**

From Eq. (3-15):

\[
\phi = \frac{TL}{GI_p} = \frac{(450 \text{ lb-in.})(9.0 \text{ in.})}{(11.4 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right)(0.5 \text{ in.})^4}
\]

\[
\phi = 0.05790 \text{ rad} = 3.32^\circ \quad \leftarrow
\]

Problem 3.3-4  An aluminum bar of solid circular cross section is twisted by torques \( T \) acting at the ends (see figure). The dimensions and shear modulus of elasticity are as follows: \( L = 1.4 \text{ m} \), \( d = 32 \text{ mm} \), and \( G = 28 \text{ GPa} \).

(a) Determine the torsional stiffness of the bar.
(b) If the angle of twist of the bar is 5°, what is the maximum shear stress? What is the maximum shear strain (in radians)?

Solution 3.3-4

(a) **Torsional stiffness of bar**

\[
d = 32 \text{ mm} \quad G = 28 \text{ GPa}
\]

\[
k_T = \frac{G I_p}{L} \quad I_p = \frac{\pi}{32} d^4
\]

\[
I_p = 1.029 \times 10^5 \text{ mm}^4
\]

\[
k_T = \frac{28(10^9)}{1.4} \left(\frac{\pi}{32} 0.032^4\right)
\]

\[
k_T = 2059 \text{ N} \cdot \text{m} \quad \leftarrow
\]

(b) **Max shear stress and strain**

\[
\phi = 5\left(\frac{\pi}{180}\right) \text{ radians}
\]

\[
T = k_T \phi \quad \tau_{\text{max}} = \frac{T\left(\frac{d}{2}\right)}{I_p}
\]

\[
\tau_{\text{max}} = 27.9 \text{ MPa} \quad \leftarrow
\]

\[
\gamma_{\text{max}} = \frac{\tau_{\text{max}}}{G}
\]

\[
\gamma_{\text{max}} = 997 \times 10^{-6} \text{ radians} \quad \leftarrow
\]
Problem 3.3-5  A high-strength steel drill rod used for boring a hole in the earth has a diameter of 0.5 in. (see figure). The allowable shear stress in the steel is 40 ksi and the shear modulus of elasticity is 11,600 ksi. What is the minimum required length of the rod so that one end of the rod can be twisted 30° with respect to the other end without exceeding the allowable stress?

Solution 3.3-5  Steel drill rod

\[ T = \frac{G\pi d^4\phi}{32L} \]

From Eq. (3-15): \( \phi = \frac{TL}{Gl_p} = \frac{32TL}{G\pi d^4} \)

\[ \tau_{\text{max}} = \frac{16}{\pi d^3} \left( \frac{G\pi d^4\phi}{32L} \right) = \frac{Gd\phi}{2L} \]

\[ L_{\text{min}} = \frac{Gd\phi}{2\tau_{\text{allow}}} = \frac{(11,600 \text{ ksi})(0.5 \text{ in.})(0.52360 \text{ rad})}{2(40 \text{ ksi})} \]

\[ L_{\text{min}} = 38.0 \text{ in.} \]

Problem 3.3-6  The steel shaft of a socket wrench has a diameter of 8.0 mm and a length of 200 mm (see figure). If the allowable stress in shear is 60 MPa, what is the maximum permissible torque \( T_{\text{max}} \) that may be exerted with the wrench? Through what angle \( \phi \) (in degrees) will the shaft twist under the action of the maximum torque? (Assume \( G = 78 \text{ GPa} \) and disregard any bending of the shaft.)
Solution 3.3-6  Socket wrench

\[ d = 8.0 \text{ mm} \quad L = 200 \text{ mm} \]
\[ \tau_{\text{allow}} = 60 \text{ MPa} \quad G = 78 \text{ GPa} \]

**Maximum permissible torque**

From Eq. (3-12): \[ \tau_{\text{max}} = \frac{16T}{\pi d^3} \]
\[ T_{\text{max}} = \frac{\pi(8.0 \text{ mm})^3(60 \text{ MPa})}{16} \]
\[ T_{\text{max}} = 6.03 \text{ N} \cdot \text{m} \leftarrow \]

**Angle of twist**

From Eq. (3-15): \[ \phi = \frac{T_{\text{max}}L}{Gl_p} \]

From Eq. (3-12): \[ T_{\text{max}} = \frac{\pi d^3 \tau_{\text{max}}}{16} \]
\[ \phi = \left( \frac{\pi d^3 \tau_{\text{max}}}{16} \right) \left( \frac{L}{Gl_p} \right) \]
\[ I_p = \frac{\pi d^4}{32} \]
\[ \phi = \frac{\pi d^3 \tau_{\text{max}}L(32)}{16G(\pi d^4)} = \frac{2\tau_{\text{max}}L}{Gd} \]
\[ \phi = \frac{2(60 \text{ MPa})(200 \text{ mm})}{(78 \text{ GPa})(8.0 \text{ mm})} = 0.03846 \text{ rad} \]
\[ \phi = (0.03846 \text{ rad}) \left( \frac{180}{\pi} \text{ deg/rad} \right) = 2.20^{\circ} \leftarrow \]

**Problem 3.3-7**  A circular tube of aluminum is subjected to torsion by torques \( T \) applied at the ends (see figure). The bar is 24 in. long, and the inside and outside diameters are 1.25 in. and 1.75 in., respectively. It is determined by measurement that the angle of twist is 4° when the torque is 6200 lb-in.

Calculate the maximum shear stress \( \tau_{\text{max}} \) in the tube, the shear modulus of elasticity \( G \), and the maximum shear strain \( \gamma_{\text{max}} \) (in radians).
Solution 3.3-7

**Numerical data**

\[
\begin{align*}
L &= 24 \text{ in.} \\
r_2 &= \frac{1.75}{2} \text{ in.} \\
r_1 &= \frac{1.25}{2} \text{ in.} \\
\phi &= 4 \left( \frac{\pi}{180} \right) \text{ radians} \\
T &= 6200 \text{ lb-in.}
\end{align*}
\]

**Max. shear stress**

\[
\tau_{\text{max}} = \frac{T r_2}{I_p}
\]

\[
I_p = \frac{\pi}{2} (r_2^4 - r_1^4) \\
I_p = 0.681 \text{ in.}^4
\]

\[
\tau_{\text{max}} = \frac{T r_2}{I_p} \quad \tau_{\text{max}} = 7965 \text{ psi}
\]

\[
\gamma_{\text{max}} = \frac{r_2}{L} \phi
\]

\[
\gamma_{\text{max}} = 0.00255 \text{ radians}
\]

**Shear modulus of elasticity**

\[
G = \frac{\tau_{\text{max}}}{\gamma_{\text{max}}}
\]

\[
G = 3.129 \times 10^6 \text{ psi}
\]

\[
G = \frac{T L}{\phi I_p} \quad G = 3.13 \times 10^6 \text{ psi}
\]

Problem 3.3-8

A propeller shaft for a small yacht is made of a solid steel bar 104 mm in diameter. The allowable stress in shear is 48 MPa, and the allowable rate of twist is 2.0° in 3.5 meters.

Assuming that the shear modulus of elasticity is \( G = 80 \text{ GPa} \), determine the maximum torque \( T_{\text{max}} \) that can be applied to the shaft.

**Solution 3.3-8**

**Numerical data**

\[
\begin{align*}
d &= 104 \text{ mm} \\
\tau_a &= 48 \text{ MPa} \\
\theta &= \frac{\phi}{L} \\
\theta &= 2 \left( \frac{\pi}{180} \right) \text{ rad/m} \\
G &= 80 \text{ GPa} \\
I_p &= \frac{\pi}{32} d^4 \\
I_p &= 1.149 \times 10^7 \text{ mm}^4
\end{align*}
\]

**Find max. torque based on allowable rate of twist**

\[
T_{\text{max}} = \frac{G L \theta}{L} \\
T_{\text{max}} = G I_p \theta
\]

\[
T_{\text{max}} = 9164 \text{ N} \cdot \text{m}
\]

\( \checkmark \text{ governs} \)

**Find max. torque based on allowable shear stress**

\[
T_{\text{max}} = \frac{\tau_a I_p}{d} \\
T_{\text{max}} = 10,602 \text{ N} \cdot \text{m}
\]
Problem 3.3-9  Three identical circular disks A, B, and C are welded to the ends of three identical solid circular bars (see figure). The bars lie in a common plane and the disks lie in planes perpendicular to the axes of the bars. The bars are welded at their intersection D to form a rigid connection. Each bar has diameter $d_1 = 0.5$ in. and each disk has diameter $d_2 = 3.0$ in.

Forces $P_1$, $P_2$, and $P_3$ act on disks A, B, and C, respectively, thus subjecting the bars to torsion. If $P_1 = 28$ lb, what is the maximum shear stress $\tau_{\text{max}}$ in any of the three bars?

Solution 3.3-9  Three circular bars

$\tau_{\text{max}} = \frac{16T}{\pi d^3} = \frac{16P_1 d_2 \sqrt{2}}{\pi d_1^3}$

$\tau_{\text{max}} = \frac{16(28 \text{ lb})(3.0 \text{ in.}) \sqrt{2}}{\pi(0.5 \text{ in.})^3} = 4840 \text{ psi}$
Problem 3.3-10  The steel axle of a large winch on an ocean liner is subjected to a torque of 1.65 kN \cdot m (see figure). What is the minimum required diameter $d_{\text{min}}$ if the allowable shear stress is 48 MPa and the allowable rate of twist is 0.75°/m? (Assume that the shear modulus of elasticity is 80 GPa.)

Solution 3.3-10

**Numerical Data**

$T = 1.65 \text{ kN} \cdot \text{m} \quad \tau_a = 48 \text{ MPa} \quad G = 80 \text{ GPa}$

$\theta_a = 0.75\left(\frac{\pi}{180}\right) \text{ rad/m}$

**Min. Required Diameter of Shaft Based on Allowable Rate of Twist**

$\theta = \frac{T}{GI_p}$

$I_p = \frac{T}{G\theta}$

$d^4 = \frac{32T}{\pi G\theta_a}$

$d_{\text{min}} = \left(\frac{32T}{\pi G\theta_a}\right)^{\frac{1}{4}}$

$\tau = \frac{Td}{2I_p}$

$\tau = \frac{Td}{2\left(\frac{\pi}{32}d^4\right)}$

$d_{\text{min}} = \left[\frac{16T}{\pi \tau_a}\right]^{\frac{1}{3}} \quad d_{\text{min}} = 0.063 \text{ m} \quad d_{\text{min}} = 63.3 \text{ mm} \quad \overset{\wedge}{\text{governs}}$

Problem 3.3-11  A hollow steel shaft used in a construction auger has outer diameter $d_2 = 6.0 \text{ in.}$ and inner diameter $d_1 = 4.5 \text{ in.}$ (see figure). The steel has shear modulus of elasticity $G = 11.0 \times 10^6 \text{ psi}$.

For an applied torque of 150 k-in., determine the following quantities:

(a) shear stress $\tau_2$ at the outer surface of the shaft,
(b) shear stress $\tau_1$ at the inner surface, and
(c) rate of twist $\theta$ (degrees per unit of length).

Also, draw a diagram showing how the shear stresses vary in magnitude along a radial line in the cross section.
Solution 3.3-11  Construction auger

\[ d_2 = 6.0 \text{ in.} \quad r_2 = 3.0 \text{ in.} \]
\[ d_1 = 4.5 \text{ in.} \quad r_1 = 2.25 \text{ in.} \]
\[ G = 11 \times 10^6 \text{ psi} \]
\[ T = 150 \text{ k-in.} \]
\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 86.98 \text{ in.}^4 \]

(a) **Shear stress at outer surface**
\[ \tau_2 = \frac{T r_2}{I_p} = \frac{(150 \text{ k-in.})(3.0 \text{ in.})}{86.98 \text{ in.}^4} = 5170 \text{ psi} \quad \leftarrow \]

(b) **Shear stress at inner surface**
\[ \tau_1 = \frac{T r_1}{I_p} = \frac{r_1}{r_2} \tau_2 = 3880 \text{ psi} \quad \leftarrow \]

(c) **Rate of twist**
\[ \theta = \frac{T}{G I_p} = \frac{(150 \text{ k-in.})}{(11 \times 10^6 \text{ psi})(86.98 \text{ in.})^4} \]
\[ \theta = 157 \times 10^{-6} \text{ rad/in.} = 0.00898 / \text{in.} \quad \leftarrow \]

(d) **Shear stress diagram**

Problem 3.3-12  Solve the preceding problem if the shaft has outer diameter \( d_2 = 150 \text{ mm} \) and inner diameter \( d_1 = 100 \text{ mm} \). Also, the steel has shear modulus of elasticity \( G = 75 \text{ GPa} \) and the applied torque is \( 16 \text{ kN} \cdot \text{m} \).

Solution 3.3-12  Construction auger

\[ d_2 = 150 \text{ mm} \quad r_2 = 75 \text{ mm} \]
\[ d_1 = 100 \text{ mm} \quad r_1 = 50 \text{ mm} \]
\[ G = 75 \text{ GPa} \]
\[ T = 16 \text{ kN} \cdot \text{m} \]
\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 39.88 \times 10^6 \text{ mm}^4 \]

(a) **Shear stress at outer surface**
\[ \tau_2 = \frac{T r_2}{I_p} = \frac{(16 \text{ kN} \cdot \text{m})(75 \text{ mm})}{39.88 \times 10^6 \text{ mm}^4} = 20.1 \text{ MPa} \quad \leftarrow \]

(b) **Shear stress at inner surface**
\[ \tau_1 = \frac{T r_1}{I_p} = \frac{r_1}{r_2} \tau_2 = 20.1 \text{ MPa} \quad \leftarrow \]

(c) **Rate of twist**
\[ \theta = \frac{T}{G I_p} = \frac{16 \text{ kN} \cdot \text{m}}{(75 \text{ GPa})(39.88 \times 10^6 \text{ mm}^4)} \]
\[ \theta = 0.005349 \text{ rad/m} = 0.306^\circ / \text{m} \quad \leftarrow \]

(d) **Shear stress diagram**

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Problem 3.3-13 A vertical pole of solid circular cross section is twisted by horizontal forces $P = 1100$ lb acting at the ends of a horizontal arm $AB$ (see figure). The distance from the outside of the pole to the line of action of each force is $c = 5.0$ in.

If the allowable shear stress in the pole is 4500 psi, what is the minimum required diameter $d_{\text{min}}$ of the pole?

Solution 3.3-13 Vertical pole

\[ P = 1100 \text{ lb} \]
\[ c = 5.0 \text{ in.} \]
\[ \tau_{\text{allow}} = 4500 \text{ psi} \]

Find $d_{\text{min}}$

\[ \tau_{\text{max}} = \frac{P(2c + d)}{\pi d^4/16} = \frac{16P(2c + d)}{\pi d^3} \]

\[ (\pi \tau_{\text{max}})d^3 - (16)Pd - 32Pc = 0 \]

Substitute numerical values:

Units: Pounds, Inches

\[(\pi)(4500)d^3 - (16)(1100)d - 32(1100)(5.0) = 0 \]

or

\[ d^3 - 1.24495d - 12.4495 = 0 \]

Solve numerically: $d = 2.496$ in.  
$d_{\text{min}} = 2.50$ in.  

Problem 3.3-14 Solve the preceding problem if the horizontal forces have magnitude $P = 5.0$ kN, the distance $c = 125$ mm, and the allowable shear stress is 30 MPa.
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Problem 3.3-15  A solid brass bar of diameter $d = 1.25$ in. is subjected to torques $T_1$, as shown in part (a) of the figure. The allowable shear stress in the brass is 12 ksi.

(a) What is the maximum permissible value of the torques $T_1$?
(b) If a hole of diameter 0.625 in. is drilled longitudinally through the bar, as shown in part (b) of the figure, what is the maximum permissible value of the torques $T_2$?
(c) What is the percent decrease in torque and the percent decrease in weight due to the hole?

Solution 3.3-14  Vertical pole

Torsion formula

$$\tau_{\text{max}} = \frac{T_r}{I_p} = \frac{Td}{2I_p}$$

$$T = P(2c + d) \quad I_p = \frac{\pi d^4}{32}$$

$$\tau_{\text{max}} = \frac{P(2c + d)d}{\pi d^4/16} = \frac{16P(2c + d)}{\pi d^3}$$

$$(\pi \tau_{\text{max}})d^3 - (16P)d - 32Pc = 0$$

Substitute numerical values:

Units: Newtons, Meters

$$(\pi)(30 \times 10^6)d^3 - (16)(5000)d - 32(5000)(0.125) = 0$$

or

$$d^3 - 848.826 \times 10^{-6}d - 212.207 \times 10^{-6} = 0$$

Solve numerically:  

\[ d = 0.06438 \text{ m} \]

\[ d_{\text{min}} = 64.4 \text{ mm} \]
Solution 3.3-15

(a) MAX. PERMISSIBLE VALUE OF TORQUE $T_1$ – SOLID BAR

$$T_{1\text{max}} = \frac{\tau_a I_p}{d} \quad T_{1\text{max}} = \frac{\tau_s \pi d^4}{32}$$

$$T_{1\text{max}} = \frac{1}{16} \tau_s \pi d^3$$

$$T_{1\text{max}} = \frac{1}{16} (12) \pi (1.25)^3$$

$$T_{1\text{max}} = 4.60 \text{ in.-k}$$

(b) MAX. PERMISSIBLE VALUE OF TORQUE $T_2$ – HOLLOW BAR

$$T_{2\text{max}} = \frac{\tau_s \pi}{32} (d_2^4 - d_1^4)$$

$$T_{2\text{max}} = \frac{d_2}{2}$$

$$T_{2\text{max}} = \frac{1}{16} \tau_s \pi \frac{d_2^4 - d_1^4}{d_2}$$

$$T_{2\text{max}} = 4.31 \text{ in.-k}$$

(c) PERCENT DECREASE IN TORQUE & PERCENT DECREASE IN WEIGHT DUE TO HOLE IN (b)

percent decrease in torque

$$\frac{T_{1\text{max}} - T_{2\text{max}}}{T_{1\text{max}}} (100) = 6.25\%$$

percent decrease in weight (weight is proportional to x-sec area)

$$A_1 = \frac{\pi}{4} d_2^2 \quad A_2 = \frac{\pi}{4} (d_2^2 - d_1^2)$$

$$\frac{A_1 - A_2}{A_1} (100) = 25\%$$

Problem 3.3-16

A hollow aluminum tube used in a roof structure has an outside diameter $d_2 = 104 \text{ mm}$ and an inside diameter $d_1 = 82 \text{ mm}$ (see figure). The tube is 2.75 m long, and the aluminum has shear modulus $G = 28 \text{ GPa}$.

(a) If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist (in degrees) when the maximum shear stress is 48 MPa?

(b) What diameter $d$ is required for a solid shaft (see figure) to resist the same torque with the same maximum stress?

(c) What is the ratio of the weight of the hollow tube to the weight of the solid shaft?
Solution 3.3-16

**Numerical data**
- \( d_2 = 104 \text{ mm} \)
- \( d_1 = 82 \text{ mm} \)
- \( L = 2.75 \times 10^3 \text{ mm} \)
- \( G = 28 \text{ GPa} \)
- \( I_p = \left(\frac{\pi}{32}\right)(d_2^4 - d_1^4) \)
- \( I_p = 7.046 \times 10^6 \text{ mm}^4 \)

(a) **Find angle of twist** \( \tau_{\text{max}} = 48 \text{ MPa} \)

\[
\phi = \frac{TL}{Gl_p} = \left(\frac{Td_2}{2I_p}\right) \frac{2L}{Gd_2}
\]

\[
\phi = \left(\frac{\tau_{\text{max}}}{Gd_2}\right) \frac{2L}{Gd_2} = 0.091 \text{ radians}
\]

\( \phi = 5.19^\circ \quad \leftarrow \)

(b) **Replace hollow shaft with solid shaft - find diameter**

\[
\tau_{\text{max}} = \frac{Td}{2 \pi} \quad \tau_{\text{max}} = \frac{16T}{d^3 \pi}
\]

set \( \tau_{\text{max}} \) expression equal to

\[
\frac{Td_2^2}{\frac{\pi}{32}(d_2^4 - d_1^4)} = \frac{32Td_2}{\pi\left(d_2^4 - d_1^4\right)}
\]

then solve for \( d \)

\[
d^3 = \frac{d_2^4 - d_1^4}{d_2}
\]

\[
d_{\text{reqd}} = \left(\frac{d_2^4 - d_1^4}{d_2}\right)^{\frac{1}{3}} = 88.4 \text{ mm} \quad \leftarrow
\]

(c) **Ratio of weights of hollow & solid shafts**

weight is proportional to cross sectional area

\[
A_h = \frac{\pi}{4}\left(d_2^2 - d_1^2\right)
\]

\[
A_s = \frac{\pi}{4}d_{\text{reqd}}^2 \quad \frac{A_h}{A_s} = 0.524 \quad \leftarrow
\]

So the weight of the tube is 52% of the solid shaft, but they resist the same torque.
Problem 3.3-17  A circular tube of inner radius $r_1$ and outer radius $r_2$ is subjected to a torque produced by forces $P = 900$ lb (see figure). The forces have their lines of action at a distance $b = 5.5$ in. from the outside of the tube.

If the allowable shear stress in the tube is 6300 psi and the inner radius $r_1 = 1.2$ in., what is the minimum permissible outer radius $r_2$?

Solution 3.3-17  Circular tube in torsion

\[
P = 900 \text{ lb}
\]
\[
b = 5.5 \text{ in.}
\]
\[
\tau_{\text{allow}} = 6300 \text{ psi}
\]
\[
r_1 = 1.2 \text{ in.}
\]

Find minimum permissible radius $r_2$

Torsion formula
\[
T = 2P(b + r_2)
\]
\[
I_p = \frac{\pi}{2}(r_2^4 - r_1^4)
\]
\[
\tau_{\text{max}} = \frac{T r_2}{I_p} = \frac{2P(b + r_2)r_2}{\frac{\pi}{2}(r_2^4 - r_1^4)} = \frac{4P(b + r_2)r_2}{\pi (r_2^4 - r_1^4)}
\]

All terms in this equation are known except $r_2$.

Solution of equation

Units: Pounds, Inches

Substitute numerical values:

\[
6300 \text{ psi} = \frac{4(900 \text{ lb})(5.5 \text{ in.} + r_2)(r_2)}{\pi [(r_2^4) - (1.2 \text{ in.})^4]}
\]
or
\[
r_2^4 - 2.07360 - 0.181891 = 0
\]
or
\[
r_2^4 - 0.181891 r_2^2 - 1.000402 r_2 - 2.07360 = 0
\]
Solve numerically:

$r_2 = 1.3988$ in.

Minimum permissible radius

$r_2 = 1.40$ in.  ←
Nonuniform Torsion

Problem 3.4-1  A stepped shaft $ABC$ consisting of two solid circular segments is subjected to torques $T_1$ and $T_2$ acting in opposite directions, as shown in the figure. The larger segment of the shaft has diameter $d_1 = 2.25$ in. and length $L_1 = 30$ in.; the smaller segment has diameter $d_2 = 1.75$ in. and length $L_2 = 20$ in. The material is steel with shear modulus $G = 11 \times 10^6$ psi, and the torques are $T_1 = 20,000$ lb-in. and $T_2 = 8,000$ lb-in.

Calculate the following quantities: (a) the maximum shear stress $\tau_{\text{max}}$ in the shaft, and (b) the angle of twist $\phi_C$ (in degrees) at end $C$.

Solution 3.4-1  Stepped shaft

Segment $AB$

$T_{AB} = T_2 - T_1 = -12,000$ lb-in.

$\tau_{AB} = \frac{16 T_{AB}}{\pi d_1^3} = \frac{16(12,000 \text{ lb-in.})}{\pi(2.25 \text{ in.})^3} = 5365$ psi

$\phi_{AB} = \frac{T_{AB}L_1}{G(I_p)_{AB}} = \frac{(-12,000 \text{ lb-in.})(30 \text{ in.})}{(11 \times 10^6 \text{ psi})(\frac{\pi}{32})(2.25 \text{ in.})^4} = -0.013007$ rad

Segment $BC$

$T_{BC} = +T_2 = 8,000$ lb-in.

$\tau_{BC} = \frac{16 T_{BC}}{\pi d_2^3} = \frac{16(8,000 \text{ lb-in.})}{\pi(1.75 \text{ in.})^3} = 7602$ psi

$\phi_{BC} = \frac{T_{BC}L_2}{G(I_p)_{BC}} = \frac{(8,000 \text{ lb-in.})(20 \text{ in.})}{(11 \times 10^6 \text{ psi})(\frac{\pi}{32})(1.75 \text{ in.})^4} = +0.015797$ rad

(a) Maximum shear stress

Segment $BC$ has the maximum stress

$\tau_{\text{max}} = 7600$ psi

(b) Angle of twist at end $C$

$\phi_C = \phi_{AB} + \phi_{BC} = (-0.013007 + 0.015797)$ rad

$\phi_C = 0.002790$ rad = $0.16^\circ$
Problem 3.4-2 A circular tube of outer diameter \(d_3 = 70\) mm and inner diameter \(d_2 = 60\) mm is welded at the right-hand end to a fixed plate and at the left-hand end to a rigid end plate (see figure). A solid circular bar of diameter \(d_1 = 40\) mm is inside of, and concentric with, the tube. The bar passes through a hole in the fixed plate and is welded to the rigid end plate.

The bar is 1.0 m long and the tube is half as long as the bar. A torque \(T = 1000\) N \(\cdot\) m acts at end \(A\) of the bar. Also, both the bar and tube are made of an aluminum alloy with shear modulus of elasticity \(G = 27\) GPa.

(a) Determine the maximum shear stresses in both the bar and tube.
(b) Determine the angle of twist (in degrees) at end \(A\) of the bar.

Solution 3.4-2 Bar and tube

Torque
\(T = 1000\) N \(\cdot\) m

(a) Maximum shear stresses

Bar: \(\tau_{\text{bar}} = \frac{16T}{\pi d_1^3} = 79.6\) MPa

Tube: \(\tau_{\text{tube}} = \frac{T(d_3/2)}{(I_p)_{\text{tube}}} = 32.3\) MPa

(b) Angle of twist at end \(A\)

Bar: \(\phi_{\text{bar}} = \frac{T L_{\text{bar}}}{G (I_p)_{\text{bar}}} = 0.1474\) rad

Tube: \(\phi_{\text{tube}} = \frac{T L_{\text{tube}}}{G (I_p)_{\text{tube}}} = 0.0171\) rad

\(\phi_A = \phi_{\text{bar}} + \phi_{\text{tube}} = 0.1474 + 0.0171 = 0.1645\) rad

\(\phi_A = 9.43^\circ\)
Problem 3.4-3  A stepped shaft $ABCD$ consisting of solid circular segments is subjected to three torques, as shown in the figure. The torques have magnitudes 12.5 k-in., 9.8 k-in., and 9.2 k-in. The length of each segment is 25 in. and the diameters of the segments are 3.5 in., 2.75 in., and 2.5 in. The material is steel with shear modulus of elasticity $G = 11.6 \times 10^3$ ksi.

(a) Calculate the maximum shear stress $\tau_{\text{max}}$ in the shaft.
(b) Calculate the angle of twist $\phi_D$ (in degrees) at end $D$.

Solution 3.4-3

Numerical data (inches, kips)

- $T_B = 12.5$ k-in.  $T_C = 9.8$ k-in.
- $T_D = 9.2$ k-in.  $L = 25$ in.
- $d_{AB} = 3.5$ in.  $d_{BC} = 2.75$ in.
- $d_{CD} = 2.5$ in.  $G = 11.6 \times (10^3)$ ksi

(a) Max. shear stress in shaft

Torque reaction at A: $R_A = -(T_B + T_C + T_D)$

$R_A = -31.5$ in.-kip

$\tau_{AB} = \frac{|R_A|d_{AB}}{2\pi d_{AB}^4}$  $\tau_{\text{max}} = 3.742$ ksi

Check CD: $\tau_{CD} = \frac{T_Dd_{CD}}{2\pi d_{CD}^4}$  $\tau_{CD} = 2.999$ ksi

Check BC: $\tau_{BC} = \frac{(T_C + T_D)d_{BC}}{2\pi d_{BC}^4}$

$\tau_{BC} = 4.65$ ksi  $\leftarrow$ controls

(b) Angle of twist at end D

$T_1 = |R_A|$  $T_2 = T_C + T_D$  $T_3 = T_D$

$I_{P1} = \frac{\pi}{32}d_{AB}^4$  $I_{P2} = \frac{\pi}{32}d_{BC}^4$

$\phi_D = \sum \frac{T_iL_i}{G I_{pi}}$  $\phi_D = \frac{L}{G} \left( \frac{T_1}{I_{P1}} + \frac{T_2}{I_{P2}} + \frac{T_3}{I_{P3}} \right)$

$\phi_D = 0.017$ radians  $\phi_D = 0.978$ degrees  $\leftarrow$
Problem 3.4-4  A solid circular bar $ABC$ consists of two segments, as shown in the figure. One segment has diameter $d_1 = 56$ mm and length $L_1 = 1.45$ m; the other segment has diameter $d_2 = 48$ mm and length $L_2 = 1.2$ m.

What is the allowable torque $T_{\text{allow}}$ if the shear stress is not to exceed 30 MPa and the angle of twist between the ends of the bar is not to exceed $1.25^\circ$? (Assume $G = 80$ GPa.)

### Solution 3.4-4

**Numerical data**

- $d_1 = 56$ mm  
- $d_2 = 48$ mm  
- $L_1 = 1450$ mm  
- $L_2 = 1200$ mm  
- $G = 80$ GPa

- $\tau_a = 30$ MPa  
- $\phi_a = 1.25\left(\frac{\pi}{180}\right)$ radians

Allowable torque

**$T_{\text{allow}}$ based on shear stress**

$$\tau_{\text{max}} = \frac{16T}{d^2\pi} \quad T_{\text{allow}} = \frac{\tau_a d^3}{16}$$

$$T_{\text{allow}} = 651.441 \text{ N} \cdot \text{m}$$

**$T_{\text{allow}}$ based on angle of twist**

$$\phi_{\text{max}} = \frac{T}{G} \left[\frac{L_1}{\frac{\pi}{32} d_1^4} + \frac{L_2}{\frac{\pi}{32} d_2^4}\right]$$

$$T_{\text{allow}} = \frac{G \phi_a}{\left(\frac{L_1}{\frac{\pi}{32} d_1^4} + \frac{L_2}{\frac{\pi}{32} d_2^4}\right)}$$

$$T_{\text{allow}} = 459 \text{ N} \cdot \text{m} \quad \leftarrow \text{ governs}$$

Problem 3.4-5  A hollow tube $ABCDE$ constructed of monel metal is subjected to five torques acting in the directions shown in the figure. The magnitudes of the torques are $T_1 = 1000$ lb-in., $T_2 = 500$ lb-in., $T_3 = 800$ lb-in., and $T_4 = 500$ lb-in., $T_5 = 800$ lb-in. The tube has an outside diameter $d_2 = 1.0$ in. The allowable shear stress is 12,000 psi and the allowable rate of twist is 2.0°/ft.

Determine the maximum permissible inside diameter $d_1$ of the tube.
Solution 3.4-5  Hollow tube of monel metal

\[ d_2 = 1.0 \text{ in.} \quad \tau_{\text{allow}} = 12,000 \text{ psi} \]
\[ \theta_{\text{allow}} = 2^\circ/\text{ft} = 0.16667^\circ/\text{in.} = 0.002909 \text{ rad/in.} \]

From Table H-2, Appendix H: \( G = 9500 \text{ ksi} \)

**Required polar moment of inertia based upon allowable shear stress**

\[ \tau_{\text{max}} = \frac{T_{\text{max}}}{I_p} \quad I_p = \frac{T_{\text{max}}(d/2)^2}{\tau_{\text{allow}}} = 0.05417 \text{ in.}^4 \]

**Required polar moment of inertia based upon allowable angle of twist**

\[ \theta = \frac{T_{\text{max}}}{GI_p} \quad I_p = \frac{T_{\text{max}}}{G\theta_{\text{allow}}} = 0.04704 \text{ in.}^4 \]

**Shear stress governs**

Required \( I_p = 0.05417 \text{ in.}^4 \)

\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) \]
\[ d_1^4 = d_3^4 - \frac{32I_p}{\pi} = (1.0 \text{ in.})^4 - \frac{32(0.05417 \text{ in.}^4)}{\pi} \]
\[ = 0.4482 \text{ in.}^4 \]
\[ d_1 = 0.818 \text{ in.} \]

(Maximum permissible inside diameter)

---

**Problem 3.4-6**  A shaft of solid circular cross section consisting of two segments is shown in the first part of the figure. The left-hand segment has diameter 80 mm and length 1.2 m; the right-hand segment has diameter 60 mm and length 0.9 m.

Shown in the second part of the figure is a hollow shaft made of the same material and having the same length. The thickness \( t \) of the hollow shaft is \( d/10 \), where \( d \) is the outer diameter. Both shafts are subjected to the same torque.

If the hollow shaft is to have the same torsional stiffness as the solid shaft, what should be its outer diameter \( d \)?
Solution 3.4-6  Solid and hollow shafts

**SOLID SHAFT CONSISTING OF TWO SEGMENTS**

\[
\phi_1 = \sum \frac{T L_i}{G I_{p_i}} = \frac{T(1.2 \text{ m})}{G \left( \frac{\pi}{32} \right) (80 \text{ mm})^4} + \frac{T(0.9 \text{ m})}{G \left( \frac{\pi}{32} \right) (60 \text{ mm})^4}
\]

\[
= \frac{32T}{\pi G} (29.297 \text{ m}^{-3} + 69.444 \text{ m}^{-3})
\]

\[
= \frac{32T}{\pi G} (98.741 \text{ m}^{-3})
\]

**HOLLOW SHAFT**

\[
d = \text{outer diameter}\]

\[
d_0 = \text{inner diameter} = 0.8d
\]

\[
\phi_2 = \frac{T L}{G I_p} = \frac{T(2.1 \text{ m})}{G \left( \frac{\pi}{32} \right) [d^4 - (0.8d)^4]}
\]

\[
= \frac{32T}{\pi G} \left( \frac{2.1 \text{ m}}{0.5904 d^4} \right) = \frac{32T}{\pi G} \left( \frac{3.5569 \text{ m}}{d^4} \right)
\]

**UNITS:** \(d = \text{meters}\)

---

**Problem 3.4-7**  Four gears are attached to a circular shaft and transmit the torques shown in the figure. The allowable shear stress in the shaft is 10,000 psi.

(a) What is the required diameter \(d\) of the shaft if it has a solid cross section?

(b) What is the required outside diameter \(d\) if the shaft is hollow with an inside diameter of 1.0 in.?
Solution 3.4-7  Shaft with four gears

\[ \tau_{\text{allow}} = 10,000 \text{ psi} \quad T_{BC} = +11,000 \text{ lb-in.} \]
\[ T_{AB} = -8000 \text{ lb-in.} \quad T_{CD} = +7000 \text{ lb-in.} \]

(a) **Solid shaft**

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} \]
\[ d^3 = \frac{16T_{\text{max}}}{\pi \tau_{\text{allow}}} = \frac{16(11,000 \text{ lb-in.})}{\pi(10,000 \text{ psi})} = 5.602 \text{ in.}^3 \]

Required \( d = 1.78 \text{ in.} \)

(b) **Hollow shaft**

Inside diameter \( d_0 = 1.0 \text{ in.} \)

\[ \tau_{\text{max}} = \frac{T_r}{I_p} \quad \tau_{\text{allow}} = \frac{T_{\text{max}}}{I_p} \]
\[ 10,000 \text{ psi} = \frac{\pi}{32} [d^4 - (1.0 \text{ in.})^4] \]

Units: \( d = \text{ inches} \)

\[ 10,000 = \frac{56,023 d}{d^4 - 1} \]

or \[ d^4 - 5.6023 d - 1 = 0 \]

Solving, \( d = 1.832 \)

Required \( d = 1.83 \text{ in.} \)

**Problem 3.4-8** A tapered bar \( AB \) of solid circular cross section is twisted by torques \( T \) (see figure). The diameter of the bar varies linearly from \( d_A \) at the left-hand end to \( d_B \) at the right-hand end.

For what ratio \( d_B/d_A \) will the angle of twist of the tapered bar be one-half the angle of twist of a prismatic bar of diameter \( d_A \)? (The prismatic bar is made of the same material, has the same length, and is subjected to the same torque as the tapered bar.) **Hint:** Use the results of Example 3-5.

**Solution 3.4-8  Tapered bar \( AB \)**

Tapered bar (From Eq. 3-27)

\[ \phi_1 = \frac{TL}{G(I_p)_A} \left( \frac{\beta^2 + \beta + 1}{3\beta^3} \right) \beta = \frac{d_B}{d_A} \]

Prismatic bar

\[ \phi_2 = \frac{TL}{G(I_p)_A} \]

Angle of twist

\[ \phi_1 = \frac{1}{2} \phi_2 \quad \frac{\beta^2 + \beta + 1}{3\beta^3} = \frac{1}{2} \]

or \[ 3\beta^3 - 2\beta^2 - 2\beta - 2 = 0 \]

Solve numerically:

\[ \beta = \frac{d_B}{d_A} = 1.45 \]
Problem 3.4-9  A tapered bar $AB$ of solid circular cross section is twisted by torques $T = 36,000$ lb-in. (see figure). The diameter of the bar varies linearly from $d_A$ at the left-hand end to $d_B$ at the right-hand end. The bar has length $L = 4.0$ ft and is made of an aluminum alloy having shear modulus of elasticity $G = 3.9 \times 10^6$ psi. The allowable shear stress in the bar is 15,000 psi and the allowable angle of twist is 3.0°.

If the diameter at end $B$ is 1.5 times the diameter at end $A$, what is the minimum required diameter $d_A$ at end $A$? (Hint: Use the results of Example 3–5).

Solution 3.4-9  Tapered bar

\[ d_B = 1.5 \, d_A \]
\[ T = 36,000 \text{ lb-in.} \]
\[ L = 4.0 \text{ ft} = 48 \text{ in.} \]
\[ G = 3.9 \times 10^6 \text{ psi} \]
\[ \tau_{\text{allow}} = 15,000 \text{ psi} \]
\[ \phi_{\text{allow}} = 3.0^\circ \]
\[ = 0.0523599 \text{ rad} \]

Minimum diameter based upon allowable shear stress

\[ \tau_{\text{max}} = \frac{16T}{\pi d_A^3} \quad d_A = \frac{16 \times 36,000 \text{ lb-in.}}{15,000 \text{ psi}} \]
\[ = 12.2231 \text{ in.}^3 \]
\[ d_A = 2.30 \text{ in.} \]

Minimum diameter based upon allowable angle of twist (From Eq. 3-27)

\[ \beta = \frac{d_B}{d_A} = 1.5 \]
\[ \phi = \frac{T L}{G(I_p)_A} \left( \frac{\beta^2 + \beta + 1}{3\beta^3} \right) = \frac{T L}{G(I_p)_A} \left( \frac{0.469136}{(36,000 \text{ lb-in.})(48 \text{ in.})} \right) \]
\[ = \frac{(3.9 \times 10^6 \text{ psi})(32)}{(36,000 \text{ lb-in.})(48 \text{ in.})} \]
\[ = 2.11728 \text{ in.}^4 \]
\[ d_A = \frac{2.11728 \text{ in.}^4}{\tau_{\text{allow}}} = \frac{2.11728 \text{ in.}^4}{0.0523599 \text{ rad}} \]
\[ = 40.4370 \text{ in.}^4 \]
\[ d_A = 2.52 \text{ in.} \]

Angle of twist governs

Min. $d_A = 2.52$ in.  \( \rightarrow \)
Problem 3.4-10  The bar shown in the figure is tapered linearly from end A to end B and has a solid circular cross section. The diameter at the smaller end of the bar is \( d_A = 25 \text{ mm} \) and the length is \( L = 300 \text{ mm} \). The bar is made of steel with shear modulus of elasticity \( G = 82 \text{ GPa} \).

If the torque \( T = 180 \text{ N} \cdot \text{m} \) and the allowable angle of twist is \( 0.3^\circ \), what is the minimum allowable diameter \( d_B \) at the larger end of the bar? ([Hint: Use the results of Example 3-5.])

Solution 3.4-10  Tapered bar

\[
\begin{align*}
d_A &= 25 \text{ mm} \\
L &= 300 \text{ mm} \\
G &= 82 \text{ GPa} \\
T &= 180 \text{ N} \cdot \text{m} \\
\phi_{\text{allow}} &= 0.3^\circ
\end{align*}
\]

Find \( d_B \)

Diameter based upon allowable angle of twist

(From Eq. 3-27)

\[
\beta = \frac{d_B}{d_A}
\]

\[
\phi = \frac{TL}{G(\rho)_A} \left( \frac{\beta^2 + \beta + 1}{3\beta^3} \right)(\rho)_A = \frac{\pi}{32}d_A
\]

\[
(0.3^\circ)\left( \frac{\pi}{180} \frac{\text{rad}}{\text{degrees}} \right) = \frac{(180 \text{ N} \cdot \text{m})(0.3 \text{ m})}{(82 \text{ GPa})\left( \frac{\pi}{32} \right)(25 \text{ mm})} \left( \frac{\beta^2 + \beta + 1}{3\beta^3} \right)
\]

\[
0.304915 = \frac{\beta^2 + \beta + 1}{3\beta^3}
\]

0.914745\( \beta^3 - \beta^2 - 1 = 0 \)

Solve numerically:

\[
\beta = 1.94452
\]

Min. \( d_B = \beta d_A = 48.6 \text{ mm} \)

\[
\rightarrow
\]
Problem 3.4-11 The nonprismatic cantilever circular bar shown has an internal cylindrical hole from 0 to \( x \), so the net polar moment of inertia of the cross section for segment 1 is \((7/8)I_p\). Torque \( T \) is applied at \( x \) and torque \( T/2 \) is applied at \( x = L \). Assume that \( G \) is constant.

(a) Find reaction moment \( R_1 \).
(b) Find internal torsional moments \( T_i \) in segments 1 & 2.
(c) Find \( x \) required to obtain twist at joint 3 of \( \varphi_3 = TL/GI_p \).
(d) What is the rotation at joint 2, \( \varphi_2 \)?
(e) Draw the torsional moment (TMD: \( T(x), 0 \leq x \leq L \)) and displacement (TDD: \( \varphi(x), 0 \leq x \leq L \)) diagrams.

Solution 3.4-11

(a) Reaction torque \( R_1 \)
\[
\sum M_x = 0 \quad R_1 = -\left( T + \frac{T}{2} \right) \quad R_1 = -\frac{3}{2} T \leftarrow
\]

(b) Internal moments in segments 1 & 2
\[
T_1 = -R_1 \quad T_1 = 1.5 T \quad T_2 = \frac{T}{2}
\]

(c) Find \( x \) required to obtain twist at joint 3
\[
\varphi_3 = \sum \frac{T_i L_i}{GI_p} \quad TL = \frac{T_1 x}{G\left(\frac{7}{8}I_p\right)} + \frac{T_2 (L - x)}{GI_p}
\]
\[
T_1 = \left(\frac{3}{2} T\right) x \quad T_2 = \left(\frac{T}{2}\right) (L - x)
\]
\[
L = \frac{17}{14} x + \frac{1}{2} L \quad x = \frac{14}{17} \left(\frac{L}{2}\right) \quad x = \frac{7}{17} L \leftarrow
\]

(d) Rotation at joint 2 for \( x \) value in (c)
\[
\varphi_2 = \frac{T_1 x}{G\left(\frac{7}{8}I_p\right)} \quad \varphi_2 = \frac{\left(\frac{3}{2} T\right) \left(\frac{7}{17} L\right)}{G\left(\frac{7}{8}I_p\right)}
\]
\[
\varphi_2 = \frac{12 T L}{17 GI_p} \quad \leftarrow
\]

(e) TMD & TDD – see plots above
TMD is constant - \( T_1 \) for 0 to \( x \) & \( T_2 \) for \( x \) to \( L \); hence TDD is linear - zero at joint 1, \( \varphi_2 \) at joint 2 & \( \varphi_3 \) at joint 3.
Problem 3.4-12  A uniformly tapered tube $AB$ of hollow circular cross section is shown in the figure. The tube has constant wall thickness $t$ and length $L$. The average diameters at the ends are $d_A$ and $d_B = 2d_A$. The polar moment of inertia may be represented by the approximate formula $I_p \approx \frac{\pi d^4 t}{4}$ (see Eq. 3-18).

Derive a formula for the angle of twist $\phi$ of the tube when it is subjected to torques $T$ acting at the ends.

\[ t = \text{thickness (constant)} \]
\[ d_A, d_B = \text{average diameters at the ends} \]
\[ d_B = 2d_A \quad I_p = \frac{\pi d^4 t}{4} \quad \text{(approximate formula)} \]

Angle of twist

Take the origin of coordinates at point $O$.

\[ d(x) = \frac{x}{2L}(d_B) = \frac{x}{L}d_A \]
\[ L_p(x) = \frac{\pi[d(x)]^3 t}{4} = \frac{\pi d_A^3}{4L^3}x^3 \]

For element of length $dx$:

\[ d\phi = \frac{Tdx}{GI_p(x)} = \frac{Tdx}{G\left(\frac{\pi d_A^4}{4L^3}\right)x^3} = \frac{4TL^3dx}{\pi Gd_A^3x^3} \]

\[ \phi = \int_L^{2L} d\phi = \frac{4TL^3}{\pi Gd_A^3} \int_L^{2L} \frac{dx}{x^3} = \frac{3TL}{2\pi Gd_A^3} \]
Problem 3.4-13  A uniformly tapered aluminum-alloy tube $AB$ of circular cross section and length $L$ is shown in the figure. The outside diameters at the ends are $d_A$ and $d_B = 2d_A$. A hollow section of length $L/2$ and constant thickness $t = d_A/10$ is cast into the tube and extends from $B$ halfway toward $A$.

(a) Find the angle of twist $\phi$ of the tube when it is subjected to torques $T$ acting at the ends. Use numerical values as follows: $d_A = 2.5$ in., $L = 48$ in., $G = 3.9 \times 10^6$ psi, and $T = 40,000$ in.-lb.

(b) Repeat (a) if the hollow section has constant diameter $d_A$. (See figure part b.)

Solution 3.4-13

Part (a) - constant thickness

use $x$ as integration variable measured from $B$ toward $A$

from B to centerline

outer and inner diameters as function of $x$

\[
0 \leq x \leq \frac{L}{2} \quad d_0(x) = d_B - \left(\frac{d_B - d_A}{L}\right)x
\]

\[
d_0(x) = 2d_A - \frac{xd_A}{L}
\]

\[
d_i(x) = (d_B - 2t) \frac{[(2d_A - 2t) - (d_A - 2t)]}{L}x
\]

\[
d_i(x) = \frac{-1}{5}d_A - \frac{9L}{5} + \frac{5x}{L}
\]

solid from centerline to $A$

\[
\frac{L}{2} \leq x \leq L \quad d_0(x) = 2d_A - \frac{xd_A}{L}
\]

\[
\phi = \frac{T}{G} \left(\frac{32}{\pi}\right) \left(\int_0^{L/2} \frac{1}{d_0^4 - d_i^4} dx + \int_{L/2}^L \frac{1}{d_i^4} dx\right)
\]
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\[ \phi = \frac{T}{G\pi} \left[ \int_0^{\frac{L}{2}} \frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4} \left(2d_A - \frac{9L + 5x}{L}\right)^4 dx + \int_{\frac{L}{2}}^L \frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4} dx \right] \]

\[ \phi = 32 \frac{T}{G\pi} \left( -\frac{125}{2} \frac{3\ln(2) + 2\ln(7) - \ln(197)}{d_A^4} L - \frac{125}{2} \frac{\ln(19) - \ln(181)}{d_A^4} L + \frac{19}{81d_A^4} L \right) \]

Simplifying: \[ \phi = \frac{16TL}{81G\pi d_A^4} \left( 38 + 10125 \ln\left(\frac{71117}{70952}\right) \right) \] or \[ \phi_a = 3.868 \frac{TL}{Gd_A^4} \]

Use numerical properties as follows \( L = 48 \text{ in.} \) \( G = 3.9 \times 10^6 \text{ psi} \) \( d_A = 2.5 \text{ in.} \) \( t = \frac{d_A}{10} \) \( T = 40000 \text{ in.-lb} \)

\( \phi_a = 0.049 \text{ radians} \) \( \phi_a = 2.79^\circ \)

**Part (b) - constant hole diameter**

\[ 0 \leq x \leq \frac{L}{2} \] \( d_0(x) = d_B - \left(\frac{d_B - d_A}{L}\right)x \)

\[ d_0(x) = 2d_A - \frac{xd_A}{L} \]

\[ \phi = \frac{T}{G\pi} \left( \int_0^{\frac{L}{2}} \frac{1}{d_0^4 - d_A^4} dx + \int_{\frac{L}{2}}^L \frac{1}{d_0^4} dx \right) \]

\[ \phi = \frac{T}{G\pi} \left[ \int_0^{\frac{L}{2}} \frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4} - d_A^4 dx + \int_{\frac{L}{2}}^L \frac{1}{\left(2d_A - \frac{xd_A}{L}\right)^4} dx \right] \]

\[ \phi_b = 32 \frac{T}{G\pi} \left( \frac{\ln(5) + 2\ln\left(\frac{3}{2}\right)}{d_A^4} - \frac{1}{4} L \frac{\ln(3) + 2\ln(2)}{d_A^4} + \frac{19}{81d_A^4} L \right) \]

Simplifying, \[ \phi_b = 3.057 \frac{TL}{Gd_A^4} \]

Use numerical properties given above \( \phi_b = 0.039 \text{ radians} \) \( \phi_b = 2.21^\circ \)

\[ \frac{\phi_a}{\phi_b} = 1.265 \text{ so tube (a) is more flexible than tube (b)} \]
Problem 3.4-14  For the thin nonprismatic steel pipe of constant thickness \( t \) and variable diameter \( d \) shown with applied torques at joints 2 and 3, determine the following.

(a) Find reaction moment \( R_1 \).
(b) Find an expression for twist rotation \( \varphi_3 \) at joint 3. Assume that \( G \) is constant.
(c) Draw the torsional moment diagram (TMD: \( T(x) \), \( 0 \leq x \leq L \)).

Solution 3.4-14

(a) **Reaction Torque** \( R_1 \)

Statics: \( \sum T = 0 \)

\[
R_1 - \frac{T}{2} + T = 0 \quad R_1 = -\frac{T}{2} \quad \leftarrow
\]

(b) **Rotation at Joint 3**

\[
d_{12}(x) = 2d \left( 1 - \frac{x}{L} \right) \quad 0 \leq x \leq \frac{L}{2}
\]

\[
d_{23}(x) = d \quad \frac{L}{2} \leq x \leq L
\]

\[
\varphi_3 = \int_0^{\frac{L}{2}} \frac{T}{G \left( \frac{4}{3} d_{12}(x)^3 t \right)} \, dx + \int_{\frac{L}{2}}^{L} \frac{T}{G \left( \frac{4}{3} d_{23}(x)^3 t \right)} \, dx
\]

use \( I_p \) expression for thin walled tubes

\[
\varphi_3 = \frac{2T}{G\pi t} \int_0^{\frac{L}{2}} \frac{1}{2d \left( 1 - \frac{x}{L} \right)} dx + \frac{4T}{G\pi d^3 t} \int_{\frac{L}{2}}^{L} dx
\]

\[
\varphi_3 = \frac{2T}{G\pi t} \int_0^{\frac{L}{2}} \frac{1}{2d \left( 1 - \frac{x}{L} \right)} dx + \frac{2TL}{G\pi d^3 t}
\]

\[
\varphi_3 = -\frac{3TL}{8G\pi d^3 t} + \frac{2TL}{G\pi d^3 t}
\]

\[
\varphi_3 = \frac{19TL}{8G\pi d^3 t} \quad \leftarrow
\]

(c) **TMD**

TMD is piecewise constant: \( T(x) = +T/2 \) for segment 1-2 & \( T(x) = +T \) for segment 2-3 (see plot above)
Problem 3.4-15 A mountain-bike rider going uphill applies torque \( T = Fd \) (\( F = 15 \text{ lb} \), \( d = 4 \text{ in.} \)) to the end of the handlebars \( ABCD \) (by pulling on the handlebar extenders \( DE \)). Consider the right half of the handlebar assembly only (assume the bars are fixed at the fork at \( A \)). Segments \( AB \) and \( CD \) are prismatic with lengths \( L_1 = 2 \text{ in.} \) and \( L_3 = 8.5 \text{ in.} \), and with outer diameters and thicknesses \( d_{01} = 1.25 \text{ in.} \), \( t_{01} = 0.125 \text{ in.} \), and \( d_{03} = 0.87 \text{ in.} \), \( t_{03} = 0.115 \text{ in.} \), respectively as shown. Segment \( BC \) of length \( L_2 = 1.2 \text{ in.} \), however, is tapered, and outer diameter and thickness vary linearly between dimensions at \( B \) and \( C \).

Consider torsion effects only. Assume \( G = 4000 \text{ ksi} \) is constant.

Derive an integral expression for the angle of twist \( \phi_D \) of half of the handlebar tube when it is subjected to torque \( T = Fd \) acting at the end. Evaluate \( \phi_D \) for the given numerical values.

Solution 3.4-15

Assume thin walled tubes

Segments \( AB \) & \( CD \)

\[
I_{p1} = \frac{\pi}{4} d_{01}^3 t_{01}, \quad I_{p3} = \frac{\pi}{4} d_{03}^3 t_{03}
\]

Segment \( BC \) \( 0 \leq x \leq L_2 \)

\[
d_{02}(x) = d_{01} \left( 1 - \frac{x}{L_2} \right) + d_{03} \left( \frac{x}{L_2} \right)
\]

\[
t_{02}(x) = t_{01} \left( 1 - \frac{x}{L_2} \right) + t_{03} \left( \frac{x}{L_2} \right)
\]

\[
\phi_D = \frac{4Fd}{G\pi} \left[ L_1 \left( \frac{d_{01}^3 t_{01}}{I_{p1}} \right) + \int_0^{L_2} \frac{L_2^4}{(d_{01} L_2 - d_{01} x + d_{03} x)^3 \times (t_{01} L_2 - t_{01} x + t_{03} x)} \, dx \right] + \frac{L_3}{d_{03}^3 t_{03}}
\]

Numerical data

\( L_1 = 2 \text{ in.} \), \( L_2 = 1.2 \text{ in.} \), \( L_3 = 8.5 \text{ in.} \),

\( t_{01} = 0.125 \text{ in.} \), \( t_{03} = 0.115 \text{ in.} \), \( d_{01} = 1.25 \text{ in.} \),

\( d_{03} = 0.87 \text{ in.} \), \( F = 15 \text{ lb} \), \( d = 4 \text{ in.} \),

\( G = 4 \times (10^6) \text{ psi} \)

\( \phi_D = 0.142^\circ \)
Problem 3.4-16 A prismatic bar $AB$ of length $L$ and solid circular cross section (diameter $d$) is loaded by a distributed torque of constant intensity $t$ per unit distance (see figure).

(a) Determine the maximum shear stress $\tau_{\text{max}}$ in the bar.
(b) Determine the angle of twist $\phi$ between the ends of the bar.

Solution 3.4-16 Bar with distributed torque

(a) Maximum shear stress

$$T_{\text{max}} = tL \quad \tau_{\text{max}} = \frac{16T_{\text{max}}}{\pi d^3} = \frac{16tL}{\pi d^3} \quad \leftarrow$$

(b) Angle of twist

$$T(x) = tx \quad I_P = \frac{\pi d^4}{32}$$

$$d\phi = T(x)dx = \frac{32txdx}{\pi Gd^4}$$

$$\phi = \int_0^L d\phi = \frac{32tx}{\pi Gd^4} \int_0^L x \, dx = \frac{16tL^2}{\pi Gd^4} \quad \leftarrow$$

Problem 3.4-17 A prismatic bar $AB$ of solid circular cross section (diameter $d$) is loaded by a distributed torque (see figure). The intensity of the torque, that is, the torque per unit distance, is denoted $t(x)$ and varies linearly from a maximum value $t_A$ at end $A$ to zero at end $B$. Also, the length of the bar is $L$ and the shear modulus of elasticity of the material is $G$.

(a) Determine the maximum shear stress $\tau_{\text{max}}$ in the bar.
(b) Determine the angle of twist $\phi$ between the ends of the bar.
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Solution 3.4-17 Bar with linearly varying torque

\[ t(x) = \frac{T_A}{L} x \]

- \( t(x) \) = intensity of distributed torque
- \( t_A \) = maximum intensity of torque
- \( d \) = diameter
- \( G \) = shear modulus
- \( T_A \) = maximum torque
  \[ T_A = \frac{1}{2} t_A L \]

(a) Maximum shear stress

\[ \tau_{\text{max}} = \frac{16T_{\text{max}}}{\pi d^3} = \frac{16T_A}{\pi d^3} = \frac{8t_A L}{\pi d^3} \]

(b) Angle of twist

\[ T(x) = \text{torque at distance } x \text{ from end } B \]

\[ T(x) = \frac{t(x) x}{2} = \frac{t_A x^2}{2L} \]

\[ I_p = \frac{\pi d^4}{32} \]

\[ d\phi = \frac{T(x) dx}{GI_p} = \frac{16t_A x^2 dx}{\pi GL^4} \]

\[ \phi = \int_0^L d\phi = \frac{16t_A}{\pi GL^4} \int_0^L x^2 dx = \frac{16t_A L^2}{3\pi G d^4} \]

Problem 3.4-18 A nonprismatic bar \( ABC \) of solid circular cross section is loaded by distributed torques (see figure).

The intensity of the torques, that is, the torque per unit distance, is denoted \( t(x) \) and varies linearly from zero at \( A \) to a maximum value \( T_0/L \) at \( B \). Segment \( BC \) has linearly distributed torque of intensity \( t(x) = T_0/3L \) of opposite sign to that applied along \( AB \). Also, the polar moment of inertia of \( AB \) is twice that of \( BC \), and the shear modulus of elasticity of the material is \( G \).

(a) Find reaction torque \( R_A \).
(b) Find internal torsional moments \( T(x) \) in segments \( AB \) and \( BC \).
(c) Find rotation \( \phi_C \).
(d) Find the maximum shear stress \( \tau_{\text{max}} \) and its location along the bar.
(e) Draw the torsional moment diagram (TMD: \( T(x) \), \( 0 \leq x \leq L \)).
Solution 3.4-18

(a) Torque reaction $R_A$

Statics: $\Sigma T = 0$

$R_A + \frac{1}{2} \left( \frac{T_0}{L} \right) \left( \frac{L}{2} \right) - \frac{1}{2} \left( \frac{T_0}{3L} \right) \left( \frac{L}{2} \right) = 0$

$R_A + \frac{T_0}{6} = 0 \quad R_A = -\frac{T_0}{6} \quad \leftarrow$

(b) Internal torsional moments in $AB$ & $BC$

$t_{AB}(x) = \frac{T_0}{6} - \left( \frac{x}{L} \right) \left( \frac{T_0}{L} \right) \left( \frac{x}{2} \right)$

$t_{AB}(x) = \left( \frac{T_0}{6} - \frac{x^2}{L^2} T_0 \right) \quad 0 \leq x \leq \frac{L}{2} \quad \leftarrow$

$t_{BC}(x) = -\left( \frac{x - L}{L} \right) \left( \frac{T_0}{2} \right)$

$t_{BC}(x) = -\left[ \left( \frac{x - L}{L} \right)^2 T_0 \right] \quad \leftarrow$

$0 \leq x \leq L$

(c) Rotation at $C$

$\phi_C = \int_0^{L/2} t_{AB}(x) dx + \int_{L/2}^L t_{BC}(x) \frac{GL_p}{I_p} dx$

$\phi_C = \int_0^{L/2} \frac{T_0}{6} - \frac{x^2}{3L^2} T_0 \frac{dx}{G(2I_p)}$

$\phi_C = \int_{L/2}^L -\left[ \left( \frac{x - L}{L} \right)^2 T_0 \right] \frac{dx}{G(2I_p)}$

$\phi_C = \frac{T_0 L}{48G(I_p)} - \frac{T_0 L}{72G(I_p)}$

$\phi_C = -\frac{T_0 L}{144G(I_p)} \quad \leftarrow$

(d) Maximum shear stress along bar

For $AB$ $2I_p = \frac{\pi}{32} d_{AB}^4$

For $BC$ $I_p = \frac{\pi}{32} d_{BC}^4$

$d_{BC} = \left( \frac{1}{2} \right) d_{AB}$

At $A$, $T = T_0/6$ $\tau_{max} = \frac{T_0}{6} \frac{d_A B}{2}$

$\tau_{max} = \frac{8T_0}{3\pi d_{AB}^3} \quad \leftarrow$ controls

Just to right of $B$, $T = -T_0/12$

$\tau_{max} = \frac{T_0}{12} \frac{d_{BC}^4}{2}$

$\tau_{max} = \frac{\pi}{32} \frac{T_0 (0.841 d_{AB})}{2}$

$\tau_{max} = \frac{\pi}{32} (0.841 d_{AB})^4$

(e) TMD = two 2nd degree curves: from $T_0/6$ at $A$, to $-T_0/12$ at $B$, to zero at $C$ (with zero slopes at $A$ & $C$ since slope on TMD is proportional to ordinate on torsional loading) – see plot of $T(x)$ above
Problem 3.4-19 A magnesium-alloy wire of diameter \( d = 4 \text{ mm} \) and length \( L \) rotates inside a flexible tube in order to open or close a switch from a remote location (see figure). A torque \( T \) is applied manually (either clockwise or counterclockwise) at end \( B \), thus twisting the wire inside the tube. At the other end \( A \), the rotation of the wire operates a handle that opens or closes the switch.

A torque \( T_0 = 0.2 \text{ N} \cdot \text{m} \) is required to operate the switch. The torsional stiffness of the tube, combined with friction between the tube and the wire, induces a distributed torque of constant intensity \( t = 0.04 \text{ N} \cdot \text{m/m} \) (torque per unit distance) acting along the entire length of the wire.

(a) If the allowable shear stress in the wire is \( \tau_{\text{allow}} = 30 \text{ MPa} \), what is the longest permissible length \( L_{\text{max}} \) of the wire?

(b) If the wire has length \( L = 4.0 \text{ m} \) and the shear modulus of elasticity for the wire is \( G = 15 \text{ GPa} \), what is the angle of twist \( \phi \) (in degrees) between the ends of the wire?

Solution 3.4-19 Wire inside a flexible tube

(a) Maximum length \( L_{\text{max}} \)

\[
\tau_{\text{allow}} = 30 \text{ MPa}
\]

Equilibrium: \( T = tL + T_0 \)

From Eq. (3-12): \( \tau_{\text{max}} = \frac{16T}{\pi d^3} \) \( T = \frac{\pi d^3 \tau_{\text{max}}}{16} \)

\[
tL + T_0 = \frac{\pi d^3 \tau_{\text{max}}}{16}
\]

\[
L = \frac{1}{16t} (\pi d^3 \tau_{\text{max}} - 16T_0)
\]

\[
L_{\text{max}} = \frac{1}{16t} (\pi d^3 \tau_{\text{allow}} - 16T_0)
\]

Substitute numerical values: \( L_{\text{max}} = 4.42 \text{ m} \)

(b) Angle of twist \( \phi \)

\[
L = 4 \text{ m} \quad G = 15 \text{ GPa}
\]

\[
\phi_1 = \text{angle of twist due to distributed torque } t
\]

\[
= \frac{16tL^2}{\pi Gd^4} \quad \text{(from problem 3.4-16)}
\]

\[
\phi_2 = \text{angle of twist due to torque } T_0
\]

\[
= \frac{T_0 L}{GIp} = \frac{32 T_0 L}{\pi Gd^4} \quad \text{(from Eq. 3-15)}
\]

\[
\phi = \text{total angle of twist}
\]

\[
= \phi_1 + \phi_2
\]

\[
\phi = \frac{16L}{\pi Gd^4} (tL + 2T_0)
\]

Substitute numerical values:

\[
\phi = 2.971 \text{ rad} = 170^\circ
\]
Problem 3.4-20  Two hollow tubes are connected by a pin at B which is inserted into a hole drilled through both tubes at B (see cross-section view at B). Tube BC fits snugly into tube AB but neglect any friction on the interface. Tube inner and outer diameters \( d_i \) \((i = 1, 2, 3)\) and pin diameter \( d_p \) are labeled in the figure. Torque \( T_0 \) is applied at joint C. The shear modulus of elasticity of the material is \( G \). Find expressions for the maximum torque \( T_{0, \text{max}} \) which can be applied at C for each of the following conditions.

(a) The shear in the connecting pin is less than some allowable value (\( \tau_{\text{pin}} < \tau_{\text{p,allow}} \)).

(b) The shear in tube AB or BC is less than some allowable value (\( \tau_{\text{tube}} < \tau_{\text{t,allow}} \)).

(c) What is the maximum rotation \( \phi_C \) for each of cases (a) and (b) above?

Solution 3.4-20

(a) \( T_{0, \text{max}} \) BASED ON ALLOWABLE SHEAR IN PIN AT B

Pin at B is in shear at interface between the two tubes; force couple \( V \cdot d_2 = T_0 \)

\[
V = \frac{T_0}{d_2}, \quad \tau_{\text{pin}} = \frac{V}{A_S}
\]

\[
\tau_{\text{pin}} = \frac{T_0}{4d_2^2}, \quad \tau_{\text{pin}} = \frac{4T_0}{\pi d_2 d_p^2}
\]

\[
T_{0, \text{max}} = \tau_{\text{p,allow}} \left( \frac{\pi d_2 d_p^2}{4} \right)
\]

(b) \( T_{0, \text{max}} \) BASED ON ALLOWABLE SHEAR IN TUBES AB & BC

\[
I_{PAB} = \frac{\pi}{32} \left( d_3^4 - d_1^4 \right)
\]

\[
I_{PAC} = \frac{\pi}{32} \left( d_2^4 - d_1^4 \right)
\]

\[
\tau_{\text{tube}} = \frac{T_0}{I_{PAB}}
\]

\[
\tau_{\text{tube}} = \frac{\pi}{32} \left( d_3^4 - d_1^4 \right)
\]

\[
\tau_{\text{tube}} = \frac{16T_0 d_3}{\pi (d_3^4 - d_1^4)}
\]

so based on tube AB:

\[
T_{0, \text{max}} = \tau_{\text{t,allow}} \left[ \frac{\pi (d_3^4 - d_1^4)}{16d_3} \right]
\]

and based on tube BC:

\[
T_{0, \text{max}} = \tau_{\text{t,allow}} \left[ \frac{\pi (d_2^4 - d_1^4)}{16d_2} \right]
\]
(c) MAX. ROTATION AT C BASED ON EITHER ALLOWABLE SHEAR IN PIN AT B OR ALLOWABLE SHEAR STRESS IN TUBES

MAX. ROTATION BASED ON ALLOWABLE SHEAR IN PIN AT B

\[ \phi_C = \frac{T_{0,\text{max}}}{G} \left( \frac{L_A}{I_{PAB}} + \frac{L_B}{I_{PBC}} \right) \]

\[ \phi_{C_{\text{max}}} = \phi_p,\text{allow} \left( \frac{\pi d^2 d_p^2}{4} \right) \]

\[ \phi_{C_{\text{max}}} = \tau_{p,\text{allow}} \left( \frac{8d^2 d_p^2}{G} \right) \]

\[ \frac{L_A}{(d_3^4 - d_2^4)} + \frac{L_B}{(d_2^4 - d_1^4)} \]

MAX. ROTATION BASED ON ALLOWABLE SHEAR STRESS IN TUBE AB

\[ \phi_{C_{\text{max}}} = \tau_{t,\text{allow}} \left( \frac{2(d_3^4 - d_2^4)}{G d_3^2} \right) \]

\[ \frac{L_A}{(d_3^4 - d_2^4)} + \frac{L_B}{(d_2^4 - d_1^4)} \]

MAX. ROTATION BASED ON ALLOWABLE SHEAR STRESS IN TUBE BC

\[ \phi_{C_{\text{max}}} = \tau_{t,\text{allow}} \left( \frac{2(d_2^4 - d_1^4)}{G d_2^2} \right) \]

\[ \frac{L_A}{(d_3^4 - d_2^4)} + \frac{L_B}{(d_2^4 - d_1^4)} \]
Pure Shear

Problem 3.5-1  A hollow aluminum shaft (see figure) has outside diameter \( d_2 = 4.0 \text{ in.} \) and inside diameter \( d_1 = 2.0 \text{ in.} \). When twisted by torques \( T \), the shaft has an angle of twist per unit distance equal to \( 0.54^\circ/\text{ft} \). The shear modulus of elasticity of the aluminum is \( G = 4.0 \times 10^6 \text{ psi} \).

(a) Determine the maximum tensile stress \( \sigma_{\text{max}} \) in the shaft.
(b) Determine the magnitude of the applied torques \( T \).

![Diagram of hollow aluminum shaft](image)

Solution 3.5-1  Hollow aluminum shaft

\( d_2 = 4.0 \text{ in.} \quad d_1 = 2.0 \text{ in.} \quad \theta = 0.54^\circ/\text{ft} \\
G = 4.0 \times 10^6 \text{ psi} \\
\)

**Maximum Shear Stress**

\[ \tau_{\text{max}} = Gr\theta \quad \text{(from Eq. 3-7a)} \]

\[ r = d_2/2 = 2.0 \text{ in.} \]

\[ \theta = (0.54^\circ/\text{ft})(\frac{1 \text{ ft}}{12 \text{ in.}})(\frac{\pi \text{ rad}}{180 \text{ degree}}) \]

\[ = 785.40 \times 10^{-6} \text{ rad/in.} \]

\[ \tau_{\text{max}} = (4.0 \times 10^6 \text{ psi})(2.0 \text{ in.})(785.40 \times 10^{-6} \text{ rad/in.}) \]

\[ = 6283.2 \text{ psi} \]

(a) **Maximum Tensile Stress**

\[ \sigma_{\text{max}} \text{ occurs on a } 45^\circ \text{ plane and is equal to } \tau_{\text{max}}. \]

\[ \sigma_{\text{max}} = \tau_{\text{max}} = 6280 \text{ psi} \]

(b) **Applied Torque**

Use the torsion formula \( \tau_{\text{max}} = \frac{Tr}{Ip} \)

\[ T = \frac{\tau_{\text{max}}Ip}{r} = \frac{\pi}{32}[(4.0 \text{ in.})^4 - (2.0 \text{ in.})^4] \]

\[ = 23.562 \text{ in.}^4 \]

\[ T = \frac{(6283.2 \text{ psi})(23.562 \text{ in.}^4)}{2.0 \text{ in.}} \]

\[ = 74,000 \text{ lb-in.} \]
Problem 3.5-2  A hollow steel bar \((G = 80 \text{ GPa})\) is twisted by torques \(T\) (see figure). The twisting of the bar produces a maximum shear strain \(\gamma_{\text{max}} = 640 \times 10^{-6} \text{ rad}\). The bar has outside and inside diameters of 150 mm and 120 mm, respectively.

(a) Determine the maximum tensile strain in the bar.
(b) Determine the maximum tensile stress in the bar.
(c) What is the magnitude of the applied torques \(T\)?

Solution 3.5-2  Hollow steel bar

\[ G = 80 \text{ GPa} \quad \gamma_{\text{max}} = 640 \times 10^{-6} \text{ rad} \]
\[ d_2 = 150 \text{ mm} \quad d_1 = 120 \text{ mm} \]
\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) \]
\[ = \frac{\pi}{32} [(150 \text{ mm})^4 - (120 \text{ mm})^4] \]
\[ = 29.343 \times 10^6 \text{ mm}^4 \]

(a) **Maximum Tensile Strain**
\[ \varepsilon_{\text{max}} = \frac{\gamma_{\text{max}}}{2} = 320 \times 10^{-6} \]

(b) **Maximum Tensile Stress**
\[ \tau_{\text{max}} = G\gamma_{\text{max}} = (80 \text{ GPa})(640 \times 10^{-6}) \]
\[ = 51.2 \text{ MPa} \]
\[ \sigma_{\text{max}} = \tau_{\text{max}} = 51.2 \text{ MPa} \]

(c) **Applied Torques**
Torsion formula: \(\tau_{\text{max}} = \frac{T\ell}{I_p} = \frac{Td_2}{2I_p}\)
\[ T = \frac{2I_p\tau_{\text{max}}}{d_2} = \frac{2(29.343 \times 10^6 \text{ mm}^4)(51.2 \text{ MPa})}{150 \text{ mm}} \]
\[ = 20,030 \text{ N} \cdot \text{m} \]
\[ = 20.0 \text{ kN} \cdot \text{m} \]

Problem 3.5-3  A tubular bar with outside diameter \(d_2 = 4.0 \text{ in.}\) is twisted by torques \(T = 70.0 \text{ k-in.}\) (see figure). Under the action of these torques, the maximum tensile stress in the bar is found to be 6400 psi.

(a) Determine the inside diameter \(d_1\) of the bar.
(b) If the bar has length \(L = 48.0 \text{ in.}\) and is made of aluminum with shear modulus \(G = 4.0 \times 10^6 \text{ psi}\), what is the angle of twist \(\phi\) (in degrees) between the ends of the bar?
(c) Determine the maximum shear strain \(\gamma_{\text{max}}\) (in radians)?
Solution 3.5-3 Tubular bar

\[ d_2 = 4.0 \text{ in.} \quad T = 70.0 \text{ k-in.} = 70,000 \text{ lb-in.} \]
\[ \sigma_{\text{max}} = 6400 \text{ psi} \quad \tau_{\text{max}} = \sigma_{\text{max}} = 6400 \text{ psi} \]

(a) Inside diameter \( d_1 \)

Torsion formula: \[ \tau_{\text{max}} = \frac{T}{I_p} = \frac{T d_2}{2 I_p} \]
\[ I_p = \frac{T d_2}{2 \tau_{\text{max}}} = \frac{(70.0 \text{ k-in.})(4.0 \text{ in.})}{2(6400 \text{ psi})} \]
\[ = 21.875 \text{ in.}^4 \]
Also, \[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = \frac{\pi}{32} [(4.0 \text{ in.})^4 - d_1^4] \]
Equate formulas:
\[ \frac{\pi}{32} [256 \text{ in.}^4 - d_1^4] = 21.875 \text{ in.}^4 \]
\[ \text{Solve for } d_1: d_1 = 2.40 \text{ in.} \]

(b) Angle of twist \( \phi \)

\[ L = 48 \text{ in.} \quad G = 4.0 \times 10^6 \text{ psi} \]
\[ \phi = \frac{T L}{G I_p} \]
From torsion formula, \[ T = \frac{2 I_p \tau_{\text{max}}}{d_2} \]
\[ \therefore \phi = \frac{2 I_p \tau_{\text{max}}}{d_2} \left( \frac{L}{G I_p} \right) = \frac{2 L \tau_{\text{max}}}{G d_2} \]
\[ = \frac{2(48 \text{ in.})(6400 \text{ psi})}{(4.0 \times 10^6 \text{ psi})(4.0 \text{ in.})} = 0.03840 \text{ rad} \]
\[ \phi = 2.20^\circ \]

(c) Maximum shear strain

\[ \gamma_{\text{max}} = \frac{\tau_{\text{max}}}{G} = \frac{6400 \text{ psi}}{4.0 \times 10^6 \text{ psi}} \]
\[ = 1600 \times 10^{-6} \text{ rad} \]

Problem 3.5-4 A solid circular bar of diameter \( d = 50 \text{ mm} \) (see figure) is twisted in a testing machine until the applied torque reaches the value \( T = 500 \text{ N·m} \). At this value of torque, a strain gage oriented at 45° to the axis of the bar gives a reading \( \varepsilon = 339 \times 10^{-6} \).

What is the shear modulus \( G \) of the material?
Solution 3.5-4  Bar in a testing machine

Strain gage at 45°:

\[ \varepsilon_{\text{max}} = 339 \times 10^{-6} \]
\[ d = 50 \text{ mm} \]
\[ T = 500 \text{ N} \cdot \text{m} \]

Shear strain (from Eq. 3-33)

\[ \gamma_{\text{max}} = 2\varepsilon_{\text{max}} = 678 \times 10^{-6} \]

Shear stress (from Eq. 3-12)

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} = \frac{16(500 \text{ N} \cdot \text{m})}{\pi(0.050 \text{ m})^3} = 20.372 \text{ MPa} \]

Shear modulus

\[ G = \frac{\tau_{\text{max}}}{\gamma_{\text{max}}} = \frac{20.372 \text{ MPa}}{678 \times 10^{-6}} = 30.0 \text{ GPa} \]

Problem 3.5-5  A steel tube \((G = 11.5 \times 10^6 \text{ psi})\) has an outer diameter \(d_2 = 2.0 \text{ in.}\) and an inner diameter \(d_1 = 1.5 \text{ in.}\).

When twisted by a torque \(T\), the tube develops a maximum normal strain of \(170 \times 10^{-6}\).

What is the magnitude of the applied torque \(T\)?

Solution 3.5-5  Steel tube

\[ G = 11.5 \times 10^6 \text{ psi} \quad d_2 = 2.0 \text{ in.} \quad d_1 = 1.5 \text{ in.} \]
\[ \varepsilon_{\text{max}} = 170 \times 10^{-6} \]

\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = \frac{\pi}{32} [(2.0 \text{ in.})^4 - (1.5 \text{ in.})^4] \]
\[ = 1.07379 \text{ in.}^4 \]

Shear strain (from Eq. 3-33)

\[ \gamma_{\text{max}} = 2\varepsilon_{\text{max}} = 340 \times 10^{-6} \]

Shear stress (from torsion formula)

\[ \tau_{\text{max}} = \frac{Tr}{I_p} = \frac{Td_2}{2I_p} \]

Also, \( \tau_{\text{max}} = G\gamma_{\text{max}} \)

Equate expressions:

\[ \frac{T d_2}{2I_p} = G\gamma_{\text{max}} \]

Solve for torque

\[ T = \frac{2G I_p \gamma_{\text{max}}}{d_2} \]
\[ = \frac{2(11.5 \times 10^6 \text{ psi})(1.07379 \text{ in.}^4)(340 \times 10^{-6})}{2.0 \text{ in.}} \]
\[ = 4200 \text{ lb-in.} \]
Problem 3.5-6  A solid circular bar of steel \((G = 78 \text{ GPa})\) transmits a torque \(T = 360 \text{ N} \cdot \text{m}\). The allowable stresses in tension, compression, and shear are 90 MPa, 70 MPa, and 40 MPa, respectively. Also, the allowable tensile strain is \(220 \times 10^{-6}\).

Determine the minimum required diameter \(d\) of the bar.

Solution 3.5-6  Solid circular bar of steel

\[ T = 360 \text{ N} \cdot \text{m} \quad G = 78 \text{ GPa} \]

**Allowable Stresses**

<table>
<thead>
<tr>
<th>Stress Type</th>
<th>Allowable Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tension</td>
<td>90 MPa</td>
</tr>
<tr>
<td>Compression</td>
<td>70 MPa</td>
</tr>
<tr>
<td>Shear</td>
<td>40 MPa</td>
</tr>
</tbody>
</table>

**Allowable Tensile Strain:**
\[ \varepsilon_{\text{max}} = 220 \times 10^{-6} \]

**Diameter Based Upon Allowable Stress**

The maximum tensile, compressive, and shear stresses in a bar in pure torsion are numerically equal. Therefore, the lowest allowable stress (shear stress) governs.

\[ \tau_{\text{allow}} = 40 \text{ MPa} \]

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} \quad d^3 = \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(360 \text{ N} \cdot \text{m})}{\pi(40 \text{ MPa})} \]

\[ d^3 = 45.837 \times 10^{-6} \text{ m}^3 \]

\[ d = 0.0358 \text{ m} = 35.8 \text{ mm} \]

Problem 3.5-7  The normal strain in the 45° direction on the surface of a circular tube (see figure) is \(880 \times 10^{-6}\) when the torque \(T = 750 \text{ lb-in}\). The tube is made of copper alloy with \(G = 6.2 \times 10^6 \text{ psi}\).

If the outside diameter \(d_2\) of the tube is 0.8 in., what is the inside diameter \(d_1\)?

Solution 3.5-7  Circular tube with strain gage

\[ d_2 = 0.80 \text{ in.} \quad T = 750 \text{ lb-in.} \quad G = 6.2 \times 10^6 \text{ psi} \]

Strain gage at 45°: \(\varepsilon_{\text{max}} = 880 \times 10^{-6}\)

**Maximum Shear Strain**
\[ \gamma_{\text{max}} = 2\varepsilon_{\text{max}} \]
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Problem 3.5-8  An aluminium tube has inside diameter \( d_1 = 50 \, \text{mm} \), shear modulus of elasticity \( G = 27 \, \text{GPa} \), and torque \( T = 4.0 \, \text{kN} \cdot \text{m} \). The allowable shear stress in the aluminum is 50 MPa and the allowable normal strain is \( 900 \times 10^{-6} \).

Determine the required outside diameter \( d_2 \).

Solution 3.5-8  Aluminum tube

**Maximum shear stress**

\[
\tau_{\text{max}} = G\gamma_{\text{max}} = 2G\varepsilon_{\text{max}}
\]

\[
\tau_{\text{max}} = \frac{T(d_2/2)}{I_p} = \frac{Td_2}{2\tau_{\text{max}}} = \frac{Td_2}{4Ge_{\text{max}}}
\]

\[
I_p = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{Td_2}{4Ge_{\text{max}}}
\]

\[
d_2^4 - d_1^4 = \frac{8Td_2}{\pi Ge_{\text{max}}} \quad d_1^4 = d_2^4 - \frac{8Td_2}{\pi Ge_{\text{max}}}
\]

**Inside diameter**

Substitute numerical values:

\[
d_2^4 = (0.8 \, \text{in.})^4 - \frac{8(750 \, \text{lb-in.})(0.80 \, \text{in.})}{\pi (6.2 \times 10^6 \, \text{psi})(880 \times 10^{-6})} = 0.4096 \, \text{in.}^4 - 0.2800 \, \text{in.}^4 = 0.12956 \, \text{in.}^4
\]

\[
d_1 = 0.60 \, \text{in.} \quad \leftarrow
\]

**Solution 3.5-8  Aluminum tube**

\[
d_1 = 50 \, \text{mm} \quad G = 27 \, \text{GPa}
\]

\[
T = 4.0 \, \text{kN} \cdot \text{m} \quad \tau_{\text{allow}} = 50 \, \text{MPa} \quad \varepsilon_{\text{allow}} = 900 \times 10^{-6}
\]

Determine the required diameter \( d_2 \).

**Allowable shear stress**

\[
(\tau_{\text{allow}})_1 = 50 \, \text{MPa}
\]

**Allowable shear stress based on normal strain**

\[
\varepsilon_{\text{max}} = \frac{\gamma}{2G} = \frac{\tau}{2G} \quad \tau = 2G\varepsilon_{\text{max}}
\]

\[
(\tau_{\text{allow}})_2 = 2G\varepsilon_{\text{allow}} = 2(27 \, \text{GPa})(900 \times 10^{-6}) = 48.6 \, \text{MPa}
\]

**Normal strain governs**

\[
\tau_{\text{allow}} = 48.60 \, \text{MPa}
\]

**Required diameter**

\[
\tau = \frac{T}{I_p} = 48.6 \, \text{MPa} = \frac{(4000 \, \text{N} \cdot \text{m})(d_2/2)}{\frac{\pi}{32}(d_2^4 - (0.050 \, \text{m})^4)}
\]

Rearrange and simplify:

\[
d_2^4 - (419.174 \times 10^{-6})d_2 - 6.25 \times 10^{-6} = 0
\]

Solve numerically:

\[
d_2 = 0.07927 \, \text{m} \quad d_2 = 79.3 \, \text{mm} \quad \leftarrow
\]
Problem 3.5-9  A solid steel bar \((G = 11.8 \times 10^6 \text{ psi})\) of
diameter \(d = 2.0\ \text{in.}\) is subjected to torques \(T = 8.0\ \text{k-in.}\) acting in the directions shown in the figure.

(a) Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.

(b) Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.

Solution 3.5-9  Solid steel bar

\[
T = 8.0\ \text{k-in.} \quad G = 11.8 \times 10^6\ \text{psi}
\]

(a) **Maximum stresses**

\[
\tau_{\text{max}} = \frac{16T}{\pi d^3} = \frac{16(8000\ \text{lb-in.})}{\pi(2.0\ \text{in.})^3} = 5093\ \text{psi} \quad \Rightarrow
\]

\[
\sigma_t = 5090\ \text{psi} \quad \sigma_c = -5090\ \text{psi} \quad \Rightarrow
\]

\[
\sigma_t = 5090\ \text{psi} \quad \sigma_c = 5090\ \text{psi}
\]

(b) **Maximum strains**

\[
\gamma_{\text{max}} = \frac{\gamma_{\text{max}}}{G} = \frac{5093\ \text{psi}}{11.8 \times 10^6\ \text{psi}} = 432 \times 10^{-6}\ \text{rad} \quad \Leftarrow
\]

\[
\varepsilon_{\text{max}} = \frac{\gamma_{\text{max}}}{2} = 216 \times 10^{-6} \quad \Rightarrow
\]

\[
\varepsilon_t = 216 \times 10^{-6} \quad \varepsilon_c = -216 \times 10^{-6} \quad \Leftarrow
\]

Problem 3.5-10  A solid aluminum bar \((G = 27\ \text{GPa})\) of
diameter \(d = 40\ \text{mm}\) is subjected to torques \(T = 300\ \text{N \cdot m}\) acting in the directions shown in the figure.

(a) Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.

(b) Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.
Transmission of Power

Problem 3.7-1  A generator shaft in a small hydroelectric plant turns at 120 rpm and delivers 50 hp (see figure).

(a) If the diameter of the shaft is \( d = 3.0 \text{ in.} \), what is the maximum shear stress \( \tau_{\text{max}} \) in the shaft?

(b) If the shear stress is limited to 4000 psi, what is the minimum permissible diameter \( d_{\text{min}} \) of the shaft?

Solution 3.7-1  Generator shaft

\[ n = 120 \text{ rpm} \quad H = 50 \text{ hp} \quad d = \text{diameter} \]

\[
\text{TORQUE} \quad H = \frac{2\pi T}{33,000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft} \\
T = \frac{33,000 H}{2\pi n} = \frac{(33,000)(50 \text{ hp})}{2\pi(120 \text{ rpm})} = 2188 \text{ lb-ft} = 26,260 \text{ lb-in.} \\
\]

(a) Maximum shear stress \( \tau_{\text{max}} \)

\[ d = 3.0 \text{ in.} \]

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} = \frac{16(26,260 \text{ lb-in.})}{\pi (3.0 \text{ in.})^3} = 4950 \text{ psi} \quad \leftarrow \]

(b) Minimum diameter \( d_{\text{min}} \)

\[ \tau_{\text{allow}} = 4000 \text{ psi} \]

\[ d_{\text{min}} = \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(26,260 \text{ lb-in.})}{\pi (4000 \text{ psi})} = 33.44 \text{ in.}^3 \]

\[ d_{\text{min}} = 3.22 \text{ in.} \quad \leftarrow \]
**Problem 3.7-2** A motor drives a shaft at 12 Hz and delivers 20 kW of power (see figure).

(a) If the shaft has a diameter of 30 mm, what is the maximum shear stress \( \tau_{\text{max}} \) in the shaft?

(b) If the maximum allowable shear stress is 40 MPa, what is the minimum permissible diameter \( d_{\text{min}} \) of the shaft?

**Solution 3.7-2** Motor-driven shaft

\[ f = 12 \text{ Hz} \quad P = 20 \text{ kW} = 20,000 \text{ N} \cdot \text{m/s} \]

**TORQUE**

\[ P = 2\pi f \cdot T \quad P = \text{watts} \quad f = \text{Hz} = s^{-1} \]

\[ T = \frac{P}{2\pi f} = \frac{20,000 \text{ W}}{2\pi(12 \text{ Hz})} = 265.3 \text{ N} \cdot \text{m} \]

(a) **MAXIMUM SHEAR STRESS** \( \tau_{\text{max}} \)

\[ d = 30 \text{ mm} \]

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} = \frac{16(265.3 \text{ N} \cdot \text{m})}{\pi(0.030 \text{ m})^3} \]

\[ = 50.0 \text{ MPa} \]

(b) **MINIMUM DIAMETER** \( d_{\text{min}} \)

\[ \tau_{\text{allow}} = 40 \text{ MPa} \]

\[ d^3 = \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(265.3 \text{ N} \cdot \text{m})}{\pi(40 \text{ MPa})} \]

\[ = 33.78 \times 10^{-6} \text{ m}^3 \]

\[ d_{\text{min}} = 0.0323 \text{ m} = 32.3 \text{ mm} \]

---

**Problem 3.7-3** The propeller shaft of a large ship has outside diameter 18 in. and inside diameter 12 in., as shown in the figure. The shaft is rated for a maximum shear stress of 4500 psi.

(a) If the shaft is turning at 100 rpm, what is the maximum horsepower that can be transmitted without exceeding the allowable stress?

(b) If the rotational speed of the shaft is doubled but the power requirements remain unchanged, what happens to the shear stress in the shaft?

**Solution 3.7-3** Hollow propeller shaft

\[ d_2 = 18 \text{ in.} \quad d_1 = 12 \text{ in.} \quad \tau_{\text{allow}} = 4500 \text{ psi} \]

\[ I_p = \frac{\pi}{32}(d_1^4 - d_2^4) = 8270.2 \text{ in.}^4 \]

**TORQUE**

\[ \tau_{\text{max}} = \frac{T(d_2/2)}{I_p} \quad T = \frac{2\tau_{\text{allow}} I_p}{d_2} \]

\[ T = \frac{2(4500 \text{ psi})(8270.2 \text{ in.}^4)}{18 \text{ in.}} \]

\[ = 4.1351 \times 10^6 \text{ lb-in.} \]

\[ = 344,590 \text{ lb-ft.} \]

(a) **HORSEPOWER**

\[ n = 100 \text{ rpm} \quad H = \frac{2\pi n T}{33,000} \]

\[ n = \text{rpm} \quad T = \text{lb-ft} \quad H = \text{hp} \]

\[ H = \frac{2\pi(100 \text{ rpm})(344,590 \text{ lb-ft})}{33,000} \]

\[ = 6560 \text{ hp} \]

---
CHAPTER 3 Torsion

Problem 3.7-5
A hollow circular shaft for use in a pumping station is being designed with an inside diameter equal to 0.75 times the outside diameter. The shaft must transmit 400 hp at 400 rpm without exceeding the allowable shear stress of 6000 psi. Determine the minimum required outside diameter $d$.

Problem 3.7-4
The drive shaft for a truck (outer diameter 60 mm and inner diameter 40 mm) is running at 2500 rpm (see figure).

(a) If the shaft transmits 150 kW, what is the maximum shear stress in the shaft?

(b) If the allowable shear stress is 30 MPa, what is the maximum power that can be transmitted?

Solution 3.7-4

**Drive shaft for a truck**

\[ d_2 = 60 \text{ mm} \quad d_1 = 40 \text{ mm} \quad n = 2500 \text{ rpm} \]

\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 1.0210 \times 10^{-6} \text{ m}^4 \]

(a) **Maximum shear stress** $\tau_{\text{max}}$

\[ P = \text{power (watts)} \quad P = 150 \text{ kW} = 150,000 \text{ W} \]

\[ T = \text{torque (newton meters)} \quad n = \text{rpm} \]

\[ P = \frac{2\pi n T}{60} \quad T = \frac{60P}{2\pi n} \]

\[ T = \frac{60(150,000 \text{ W})}{2\pi(2500 \text{ rpm})} = 572.96 \text{ N} \cdot \text{m} \]

\[ \tau_{\text{max}} = \frac{T d_2^2}{2 I_p} = \frac{(572.96 \text{ N} \cdot \text{m})(0.060 \text{ m})}{2(1.0210 \times 10^{-6} \text{ m}^4)} \]

\[ = 16.835 \text{ MPa} \]

\[ \tau_{\text{max}} = 16.8 \text{ MPa} \quad \leftarrow \]

(b) **Maximum power** $P_{\text{max}}$

\[ \tau_{\text{allow}} = 30 \text{ MPa} \]

\[ P_{\text{max}} = P \frac{\tau_{\text{allow}}}{\tau_{\text{max}}} = (150 \text{ kW}) \left( \frac{30 \text{ MPa}}{16.835 \text{ MPa}} \right) \]

\[ = 267 \text{ kW} \quad \leftarrow \]
Solution 3.7-5  Hollow shaft

\[ d = \text{outside diameter} \]
\[ d_0 = \text{inside diameter} \]
\[ = 0.75 \, d \]
\[ H = 400 \, \text{hp} \quad n = 400 \, \text{rpm} \]
\[ \tau_{\text{allow}} = 6000 \, \text{psi} \]
\[ I_p = \frac{\pi}{32} [d^4 - (0.75 \, d)^4] = 0.067112 \, d^4 \]

Torque
\[ H = \frac{2 \pi n T}{33,000} \]
\[ T = \frac{33,000 \, H}{2 \pi n} = \frac{(33,000)(400 \, \text{hp})}{2 \pi(400 \, \text{rpm})} = 5252.1 \, \text{lb-ft} = 63,025 \, \text{lb-in.} \]

Minimum outside diameter
\[ \tau_{\text{max}} = \frac{T d}{2 I_p} \]
\[ I_p = \frac{T d}{2 \tau_{\text{max}}} = \frac{T d}{2 \tau_{\text{allow}}} \]
\[ 0.067112 \, d^4 = \frac{(63,025 \, \text{lb-in.})(d)}{2(6000 \, \text{psi})} \]
\[ d^3 = 78.259 \, \text{in.}^3 \quad d_{\text{min}} = 4.28 \, \text{in.} \quad \leftarrow \]

Problem 3.7-6  A tubular shaft being designed for use on a construction site must transmit 120 kW at 1.75 Hz. The inside diameter of the shaft is to be one-half of the outside diameter.

If the allowable shear stress in the shaft is 45 MPa, what is the minimum required outside diameter \( d \)?

Solution 3.7-6  Tubular shaft

\[ d = \text{outside diameter} \]
\[ d_0 = \text{inside diameter} \]
\[ = 0.5 \, d \]
\[ P = 120 \, \text{kW} = 120,000 \, \text{W} \quad f = 1.75 \, \text{Hz} \]
\[ \tau_{\text{allow}} = 45 \, \text{MPa} \]
\[ I_p = \frac{\pi}{32} [d^4 - (0.5 \, d)^4] = 0.092039 \, d^4 \]

Torque
\[ P = 2 \pi f T \quad P = \text{watts} \quad f = \text{Hz} \]
\[ T = \text{newton meters} \]
\[ T = \frac{P}{2 \pi f} = \frac{120,000 \, \text{W}}{2 \pi(1.75 \, \text{Hz})} = 10,913.5 \, \text{N} \cdot \text{m} \]

Minimum outside diameter
\[ \tau_{\text{max}} = \frac{T d}{2 I_p} \]
\[ I_p = \frac{T d}{2 \tau_{\text{max}}} = \frac{T d}{2 \tau_{\text{allow}}} \]
\[ 0.092039 \, d^4 = \frac{(10,913.5 \, \text{N} \cdot \text{m})(d)}{2(45 \, \text{MPa})} \]
\[ d^3 = 0.0013175 \, \text{m}^3 \quad d = 0.1096 \, \text{m} \]
\[ d_{\text{min}} = 110 \, \text{mm} \quad \leftarrow \]

Problem 3.7-7  A propeller shaft of solid circular cross section and diameter \( d \) is spliced by a collar of the same material (see figure). The collar is securely bonded to both parts of the shaft.

What should be the minimum outer diameter \( d_1 \) of the collar in order that the splice can transmit the same power as the solid shaft?
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Solution 3.7-7  Splice in a propeller shaft

**SOLID SHAFT**

\[ \tau_{\text{max}} = \frac{16 T_1}{\pi d^3}, \quad T_1 = \frac{\pi d^3 \tau_{\text{max}}}{16} \]

**HOLLOW COLLAR**

\[ I_p = \frac{\pi}{32} (d_1^4 - d^4), \quad \tau_{\text{max}} = \frac{T_2 r}{I_p} = \frac{T_2 (d_1/2)}{I_p} \]

\[ T_2 = 2 \frac{\tau_{\text{max}} I_p}{d_1} = \frac{2\pi}{\pi} \left( \frac{\pi}{32} \right) (d_1^4 - d^4) \]

\[ = \frac{\pi \tau_{\text{max}}}{16} \left( d_1^3 - d^4 \right) \]

**EQUATE TORQUES**

For the same power, the torques must be the same. For the same material, both parts can be stressed to the same maximum stress.

\[ T_1 = T_2 \cdot \frac{\pi d^3 \tau_{\text{max}}}{16} = \frac{\pi \tau_{\text{max}}}{16d_1} (d_1^4 - d^4) \]

\[ \text{or} \quad \left( \frac{d_1}{d} \right)^4 - \frac{d_1}{d} - 1 = 0 \quad \text{(Eq. 1)} \]

**MINIMUM OUTER DIAMETER**

Solve Eq. (1) numerically:

Min. \( d_1 = 1.221 \) \( d \)

---

**Problem 3.7-8**  What is the maximum power that can be delivered by a hollow propeller shaft (outside diameter 50 mm, inside diameter 40 mm, and shear modulus of elasticity 80 GPa) turning at 600 rpm if the allowable shear stress is 100 MPa and the allowable rate of twist is 3.0°/m?

**Solution 3.7-8  Hollow propeller shaft**

\( d_2 = 50 \text{ mm} \quad d_1 = 40 \text{ mm} \)

\( G = 80 \text{ GPa} \quad n = 600 \text{ rpm} \)

\( \tau_{\text{allow}} = 100 \text{ MPa} \quad \theta_{\text{allow}} = 3.0^\circ/\text{m} \)

\[ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 362.3 \times 10^{-9} \text{ m}^4 \]

**BASED UPON ALLOWABLE SHEAR STRESS**

\[ \tau_{\text{max}} = \frac{T_1 (d_2/2)}{I_p} \quad T_1 = \frac{2\tau_{\text{allow}} I_p}{d_2} \]

\[ T_1 = \frac{2(100 \text{ MPa})(362.3 \times 10^{-9} \text{ m}^4)}{0.050 \text{ m}} \]

\[ = 1449 \text{ N} \cdot \text{m} \]

**BASED UPON ALLOWABLE RATE OF TWIST**

\[ \theta = \frac{T_2}{G I_p} \quad T_2 = G I_p \theta_{\text{allow}} \]

\[ T_2 = (80 \text{ GPa}) (362.3 \times 10^{-9} \text{ m}^4) (3.0^\circ/\text{m}) \times \left( \frac{\pi}{180} \text{ rad/degree} \right) \]

\[ T_2 = 1517 \text{ N} \cdot \text{m} \]

**SHEAR STRESS GOVERNS**

\( T_{\text{allow}} = T_1 = 1449 \text{ N} \cdot \text{m} \)

**MAXIMUM POWER**

\[ P = \frac{2\pi n T}{60} = \frac{2\pi(600 \text{ rpm})(1449 \text{ N} \cdot \text{m})}{60} \]

\[ P = 91,047 \text{ W} \]

\( P_{\text{max}} = 91.0 \text{ kW} \)
Problem 3.7-9 A motor delivers 275 hp at 1000 rpm to the end of a shaft (see figure). The gears at B and C take out 125 and 150 hp, respectively.

Determine the required diameter \( d \) of the shaft if the allowable shear stress is 7500 psi and the angle of twist between the motor and gear C is limited to 1.5°. (Assume \( G = 11.5 \times 10^6 \) psi, \( L_1 = 6 \) ft, and \( L_2 = 4 \) ft.)

![Diagram of motor-driven shaft](image)

**Solution 3.7-9 Motor-driven shaft**

\( L_1 = 6 \) ft  
\( L_2 = 4 \) ft  
\( d = \) diameter  
\( n = 1000 \) rpm  
\( \tau_{\text{allow}} = 7500 \) psi  
\( (\phi_{AC})_{\text{allow}} = 1.5^\circ = 0.02618 \) rad  
\( G = 11.5 \times 10^6 \) psi

**Torques acting on the shaft**

\[
H = \frac{2\pi n T}{33,000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}
\]

\[
T = \frac{33,000 H}{2\pi n}
\]

At point A: \( T_A = \frac{33,000(275 \text{ hp})}{2\pi(1000 \text{ rpm})} \)

\[
= 1444 \text{ lb-ft}
\]

\[
= 17,332 \text{ lb-in.}
\]

At point B: \( T_B = \frac{125}{275} T_A = 7878 \text{ lb-in.} \)

At point C: \( T_C = \frac{150}{275} T_A = 9454 \text{ lb-in.} \)

**Free-body diagram**

\( T_A = 17,332 \text{ lb-in.} \)

\( T_C = 9454 \text{ lb-in.} \)

\( T_B = 7878 \text{ lb-in.} \)

**Internal torques**

\( T_{AB} = 17,332 \text{ lb-in.} \)

\( T_{BC} = 9454 \text{ lb-in.} \)

**Diameter based upon allowable shear stress**

The larger torque occurs in segment AB

\[
\tau_{\text{max}} = \frac{16 T_{AB}}{\pi d^3} \quad d^3 = \frac{16 T_{AB}}{\pi \tau_{\text{allow}}}
\]

\[
= \frac{16(17,332 \text{ lb-in.})}{\pi(7500 \text{ psi})} = 11.77 \text{ in.}^3
\]

\( d = 2.27 \) in.

**Diameter based upon allowable angle of twist**

\[
I_p = \frac{\pi d^4}{32} \quad \phi = \frac{TL}{G I_p} = \frac{32TL}{\pi G d^4}
\]
Problem 3.7-10  The shaft $ABC$ shown in the figure is driven by a motor that delivers 300 kW at a rotational speed of 32 Hz. The gears at $B$ and $C$ take out 120 and 180 kW, respectively. The lengths of the two parts of the shaft are $L_1 = 1.5$ m and $L_2 = 0.9$ m.

Determine the required diameter $d$ of the shaft if the allowable shear stress is 50 MPa, the allowable angle of twist between points $A$ and $C$ is 4.0°, and $G = 75$ GPa.

Solution 3.7-10  Motor-driven shaft

At point $A$: $T_A = \frac{300,000 \text{ W}}{2\pi(32 \text{ Hz})} = 1492 \text{ N} \cdot \text{m}$

At point $B$: $T_B = \frac{120}{300} T_A = 596.8 \text{ N} \cdot \text{m}$

At point $C$: $T_C = \frac{180}{300} T_A = 895.3 \text{ N} \cdot \text{m}$

Free-body diagram

$T_A = 1492 \text{ N} \cdot \text{m}$

$T_B = 596.8 \text{ N} \cdot \text{m}$

$T_C = 895.3 \text{ N} \cdot \text{m}$

$d = \text{diameter}$
SECTION 3.7 Transmission of Power

INTERNAL TORQUES

\[ T_{AB} = 1492 \text{ N} \cdot \text{m} \]
\[ T_{BC} = 895.3 \text{ N} \cdot \text{m} \]

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment \( AB \)

\[ \tau_{\text{max}} = \frac{16 T_{AB}}{\pi d^3} \cdot d^3 = \frac{16 T_{AB}}{\pi \tau_{\text{allow}}} = \frac{16(1492 \text{ N} \cdot \text{m})}{\pi(50 \text{ MPa})} \]
\[ d^3 = 0.0001520 \text{ m}^3 \quad d = 0.0534 \text{ m} = 53.4 \text{ mm} \]

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

\[ I_p = \frac{\pi d^4}{32} \quad \phi = \frac{TL}{GI_p} = \frac{32TL}{\pi Gd^4} \]

Segment \( AB \):

\[ \phi_{AB} = \frac{32 T_{AB}L_{AB}}{\pi Gd^4} = \frac{32(1492 \text{ N} \cdot \text{m})(1.5 \text{ m})}{\pi(75 \text{ GPa})d^4} \]
\[ \phi_{AB} = \frac{0.3039 \times 10^{-6}}{d^4} \]

Segment \( BC \):

\[ \phi_{BC} = \frac{32 T_{BC}L_{BC}}{\pi Gd^4} = \frac{32(895.3 \text{ N} \cdot \text{m})(0.9 \text{ m})}{\pi(75 \text{ GPa})d^4} \]
\[ \phi_{BC} = \frac{0.1094 \times 10^{-6}}{d^4} \]

From \( A \) to \( C \): \( \phi_{AC} = \phi_{AB} + \phi_{BC} = \frac{0.4133 \times 10^{-6}}{d^4} \)

\( (\phi_{AC})_{\text{allow}} = 0.06981 \text{ rad} \)

\[ \therefore 0.06981 = \frac{0.1094 \times 10^{-6}}{d^4} \]

and \( d = 0.04933 \text{ m} = 49.3 \text{ mm} \)

Shear stress governs

\( d = 53.4 \text{ mm} \)
Statically Indeterminate Torsional Members

Problem 3.8-1 A solid circular bar $ABCD$ with fixed supports is acted upon by torques $T_0$ and $2T_0$ at the locations shown in the figure. Obtain a formula for the maximum angle of twist $\phi_{\text{max}}$ of the bar. (Hint: Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)

Solution 3.8-1 Circular bar with fixed ends

From Eqs. (3-46a and b):

$$T_A = \frac{T_0 L_B}{L}$$
$$T_B = \frac{T_0 L_A}{L}$$

Apply the above formulas to the given bar:

$$T_A = T_0 \left( \frac{7}{10} \right) + 2T_0 \left( \frac{4}{10} \right) = \frac{15T_0}{10}$$
$$T_D = T_0 \left( \frac{3}{10} \right) + 2T_0 \left( \frac{6}{10} \right) = \frac{15T_0}{10}$$

Problem 3.8-2 A solid circular bar $ABCD$ with fixed supports at ends $A$ and $D$ is acted upon by two equal and oppositely directed torques $T_0$, as shown in the figure. The torques are applied at points $B$ and $C$, each of which is located at distance $x$ from one end of the bar. (The distance $x$ may vary from zero to $L/2$.)

(a) For what distance $x$ will the angle of twist at points $B$ and $C$ be a maximum?
(b) What is the corresponding angle of twist $\phi_{\text{max}}$?

(Hint: Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)
Solution 3.8-2  Circular bar with fixed ends

From Eqs. (3-46a and b):

\[ T_A = \frac{T_0 L_B}{L} \]
\[ T_B = \frac{T_0 L_A}{L} \]

Apply the above formulas to the given bar:

\[ T_A = \frac{T_0 (L - x)}{L} - \frac{T_0 x}{L} = \frac{T_0}{L} (L - 2x) \quad T_D = T_A \]

Problem 3.8-3  A solid circular shaft \( AB \) of diameter \( d \) is fixed against rotation at both ends (see figure). A circular disk is attached to the shaft at the location shown.

What is the largest permissible angle of rotation \( \phi_{max} \) of the disk if the allowable shear stress in the shaft is \( \tau_{allow} \)? (Assume that \( a > b \). Also, use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)

Solution 3.8-3  Shaft fixed at both ends

Assume that a torque \( T_0 \) acts at the disk.

The reactive torques can be obtained from Eqs. (3-46a and b):

\[ T_A = \frac{T_0 b}{L} \quad T_B = \frac{T_0 a}{L} \]

Since \( a > b \), the larger torque (and hence the larger stress) is in the right hand segment.
Problem 3.8-4 A hollow steel shaft $ACB$ of outside diameter 50 mm and inside diameter 40 mm is held against rotation at ends $A$ and $B$ (see figure). Horizontal forces $P$ are applied at the ends of a vertical arm that is welded to the shaft at point $C$.

Determine the allowable value of the forces $P$ if the maximum permissible shear stress in the shaft is 45 MPa. (Hint: Use Eqs. 3-46a and b of Example 3-9 to obtain the reactive torques.)

Solution 3.8-4 Hollow shaft with fixed ends

General Formulas:

Apply the above formulas to the given shaft

The larger torque, and hence the larger shear stress, occurs in part $CB$ of the shaft.
Problem 3.8-5 A stepped shaft $ACB$ having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

If the allowable shear stress in the shaft is 6000 psi, what is the maximum torque ($T_{0\text{max}}$) that may be applied at section $C$? (Hint: Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)

Solution 3.8-5 Stepped shaft $ACB$

$$\tau_{\text{max}} = \frac{T_{\text{max}}(d/2)}{I_p} \quad T_{\text{max}} = \frac{2\tau_{\text{max}}I_p}{d} \quad \text{(Eq. 1)}$$

Units: Newtons and meters

$$\tau_{\text{max}} = 45 \times 10^6 \text{N/m}^2$$

$$I_p = \frac{\pi}{32} (d^4_2 - d^4_1) = 362.26 \times 10^{-9} \text{m}^4$$

$d = d_2 = 0.05 \text{ mm}$

Substitute numerical values into (Eq. 1):

$$P_{\text{allow}} = \frac{2(45 \times 10^6 \text{N/m}^2)(362.26 \times 10^{-9} \text{m}^4)}{0.05 \text{ m}} = 652.07 \text{ N \cdot m}$$

$P = \frac{652.07 \text{ N \cdot m}}{0.24 \text{ m}} = 2717 \text{ N}$

$P_{\text{allow}} = 2720 \text{ N}$

$0.75 \text{ in.}$

$1.50 \text{ in.}$

$6.0 \text{ in.}$

$15.0 \text{ in.}$

Combine Eqs. (1) and (3) and solve for $T_{0\text{AC}}$:

$$T_{0\text{AC}} = \frac{1}{16} \pi d^3_0 \tau_{\text{allow}} \left( 1 + \frac{L_AL_B}{L_B d_A + L_A d_B} \right)$$

Substitute numerical values:

$(T_{0\text{AC}}) = 3678 \text{ lb-in.}$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT $CB$

$$\tau_{\text{CB}} = \frac{16T_B}{\pi d^3_B}$$

$$T_B = \frac{1}{16} \pi d^3_B \tau_{\text{CB}} = \frac{1}{16} \pi d^3_B \tau_{\text{allow}}$$

$$d_A = 0.75 \text{ in.} \quad d_B = 1.50 \text{ in.}$$

$L_A = 6.0 \text{ in.} \quad L_B = 15.0 \text{ in.}$

$\tau_{\text{allow}} = 6000 \text{ psi}$

Find $(T_{0\text{max}})$

REACTIVE TORQUES (from Eqs. 3-45a and b)

$$T_A = T_0 \left( \frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}} \right) \quad \text{(1)}$$

$$T_B = T_0 \left( \frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}} \right) \quad \text{(2)}$$

ALLOWABLE TORQUE BASED UPON SHEAR STRESS IN SEGMENT $AC$

$$\tau_{AC} = \frac{16T_A}{\pi d^3_A}$$

$$T_A = \frac{1}{16} \pi d^3_A \tau_{AC} = \frac{1}{16} \pi d^3_A \tau_{\text{allow}} \quad \text{(3)}$$

$0.24 = \frac{16 \times 6000 \text{ N/m}^2}{\pi \times (0.05 \text{ mm})^3}$

$.24 = \frac{16 \times 6000 \text{ N/m}^2}{\pi \times (0.05 \text{ mm})^3}$
Combine Eqs. (2) and (5) and solve for $T_0$:

$$ (T_0)_{CB} = \frac{1}{16} \pi d_B^3 \tau_{\text{allow}} \left( 1 + \frac{L_B d_{PA}}{L_A d_{PB}} \right) $$

$$ = \frac{1}{16} \pi d_B^3 \tau_{\text{allow}} \left( 1 + \frac{L_B d_A^4}{L_A d_B^4} \right) $$

(6)

Substitute numerical values:

$$ (T_0)_{CB} = 4597 \text{ lb-in.} $$

Segment AC governs

$$(T_0)_{\max} = 3680 \text{ lb-in.} \quad \leftarrow$$

NOTE: From Eqs. (4) and (6) we find that

$$ \frac{(T_0)_{AC}}{(T_0)_{CB}} = \left( \frac{L_A}{L_B} \right) \left( \frac{d_B}{d_A} \right) $$

which can be used as a partial check on the results.

**Problem 3.8-6** A stepped shaft $ACB$ having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

If the allowable shear stress in the shaft is 43 MPa, what is the maximum torque $(T_0)_{\max}$ that may be applied at section $C$?

(Hint: Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)

**Solution 3.8-6 Stepped shaft $ACB$**

$$d_A = 20 \text{ mm}$$
$$d_B = 25 \text{ mm}$$
$$L_A = 225 \text{ mm}$$
$$L_B = 450 \text{ mm}$$
$$\tau_{\text{allow}} = 43 \text{ MPa}$$

Find $(T_0)_{\max}$

**REACTIVE TORQUES** (from Eqs. 3-45a and b)

$$T_A = T_0 \left( \frac{L_B d_{PA}}{L_B d_{PA} + L_A d_{PB}} \right) \quad (1)$$

$$T_B = T_0 \left( \frac{L_A d_{PB}}{L_B d_{PA} + L_A d_{PB}} \right) \quad (2)$$

**ALLOWABLE TORQUE BASED UPON SHEAR STRESS in segment AC**

$$\tau_{AC} = \frac{16T_A}{\pi d_A^3}$$

Combine Eqs. (1) and (3) and solve for $T_0$:

$$T_A = \frac{1}{16} \pi d_A^3 \tau_{AC} = \frac{1}{16} \pi d_A^3 \tau_{\text{allow}} \quad (3)$$

Substitute numerical values:

$$(T_0)_{AC} = 150.0 \text{ N} \cdot \text{m}$$

**ALLOWABLE TORQUE BASED UPON SHEAR STRESS in segment CB**

$$\tau_{CB} = \frac{16T_B}{\pi d_B^3} $$

$$T_B = \frac{1}{16} \pi d_B^3 \tau_{CB} = \frac{1}{16} \pi d_B^3 \tau_{\text{allow}} \quad (5)$$
SECTION 3.8 Statically Indeterminate Torsional Members

Problem 3.8-7 A stepped shaft \( ACB \) is held against rotation at ends \( A \) and \( B \) and subjected to a torque \( T_0 \) acting at section \( C \) (see figure). The two segments of the shaft (\( AC \) and \( CB \)) have diameters \( d_A \) and \( d_B \), respectively, and polar moments of inertia \( I_{PA} \) and \( I_{PB} \), respectively. The shaft has length \( L \) and segment \( AC \) has length \( a \).

(a) For what ratio \( a/L \) will the maximum shear stresses be the same in both segments of the shaft?
(b) For what ratio \( a/L \) will the internal torques be the same in both segments of the shaft? (Hint: Use Eqs. 3-45a and b of Example 3-9 to obtain the reactive torques.)

Solution 3.8-7 Stepped shaft

SEGMENT AC: \( d_A, I_{PA}, L_A = a \)
SEGMENT CB: \( d_B, I_{PB}, L_B = L - a \)
REACTIVE TORQUES (from Eqs. 3-45a and b)
\[
T_A = T_0 \left( \frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}} \right) ; \quad T_B = T_0 \left( \frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}} \right)
\]
(a) \( \text{EQUAL SHEAR STRESSES} \)
\[
\tau_{AC} = \frac{T_A (d_A/2)}{I_{PA}} = \frac{T_B (d_B/2)}{I_{PB}}
\]
\[
\tau_{AC} = \tau_{CB} \quad \text{or} \quad \frac{T_A d_A}{I_{PA}} = \frac{T_B d_B}{I_{PB}} \quad (\text{Eq. 1})
\]
Substitute \( T_A \) and \( T_B \) into Eq. (1):
\[
\frac{L_B I_{PA} d_A}{I_{PA}} = \frac{L_A I_{PB} d_B}{I_{PB}} \quad \text{or} \quad L_B d_A = L_A d_B
\]

(b) \( \text{EQUAL TORQUES} \)
\[
T_A = T_B \quad \text{or} \quad L_B I_{PA} = L_A I_{PB}
\]
\[
\text{or} \quad (L - a) I_{PA} = a I_{PB}
\]
Solve for \( a/L \):
\[
\frac{a}{L} = \frac{I_{PA}}{I_{PA} + I_{PB}} \quad \text{or} \quad \frac{a}{L} = \frac{d_A^4}{d_A^4 + d_B^4} \quad \text{←}
\]
Problem 3.8-8  A circular bar $AB$ of length $L$ is fixed against rotation at the ends and loaded by a distributed torque $t(x)$ that varies linearly in intensity from zero at end $A$ to $t_0$ at end $B$ (see figure).

Obtain formulas for the fixed-end torques $T_A$ and $T_B$.

Solution 3.8-8  Fixed-end bar with triangular load

$t(x) = \frac{t_0 x}{L}$

$T_0 = \text{Resultant of distributed torque}$

$T_0 = \int_0^L t(x) dx = \int_0^L \frac{t_0 x}{L} dx = \frac{t_0 L}{2}$

Equilibrium

$T_A + T_B = T_0 = \frac{t_0 L}{2}$

Problem 3.8-9  A circular bar $AB$ with ends fixed against rotation has a hole extending for half of its length (see figure). The outer diameter of the bar is $d_2 = 3.0$ in. and the diameter of the hole is $d_1 = 2.4$ in. The total length of the bar is $L = 50$ in.

At what distance $x$ from the left-hand end of the bar should a torque $T_0$ be applied so that the reactive torques at the supports will be equal?
Solution 3.8-9  Bar with a hole

\[ L = 50 \text{ in.} \]
\[ L/2 = 25 \text{ in.} \]
\[ d_2 = \text{outer diameter} = 3.0 \text{ in.} \]
\[ d_1 = \text{diameter of hole} = 2.4 \text{ in.} \]
\[ T_0 = \text{Torque applied at distance } x \]

Find \( x \) so that \( T_A = T_B \)

**EQUILIBRIUM**

\[ T_A + T_B = T_0 \quad \therefore \quad T_A = T_B = \frac{T_0}{2} \quad (1) \]

**REMOVE THE SUPPORT AT END B**

\[ \phi_B = \text{Angle of twist at } B \]

\[ I_{PA} = \text{Polar moment of inertia at left-hand end} \]
\[ I_{PB} = \text{Polar moment of inertia at right-hand end} \]

\[ \phi_B = \frac{T_0}{G I_{PB}} + \frac{T_0(L/2)}{G I_{PA}} - \frac{T_0(x - L/2)}{G I_{PB}} \]

Substitute Eq. (1) into Eq. (2) and simplify:

\[ \phi_B = \frac{T_0}{G} \left[ \frac{L}{4I_{PB}} + \frac{L}{4I_{PA}} - \frac{x}{I_{PB}} + \frac{L}{2I_{PB}} - \frac{L}{2I_{PA}} \right] \]

**COMPATIBILITY**

\[ \phi_B = 0 \]

\[ \therefore \quad \frac{x}{I_{PB}} = \frac{3L}{4I_{PB}} - \frac{L}{4I_{PA}} \]

Solve for \( x \):

\[ x = \frac{L}{4} \left( 3 - \frac{I_{PB}}{I_{PA}} \right) \]

\[ \frac{I_{PB}}{I_{PA}} = \frac{d_2^4 - d_1^4}{d_1^4} = 1 - \left( \frac{d_1}{d_2} \right)^4 \]

\[ x = \frac{L}{4} \left[ 2 + \left( \frac{d_1}{d_2} \right)^4 \right] \]

SUBSTITUTE NUMERICAL VALUES:

\[ x = \frac{50 \text{ in.}}{4} \left[ 2 + \left( \frac{2.4 \text{ in.}}{3.0 \text{ in.}} \right)^4 \right] = 30.12 \text{ in.} \]

Problem 3.8-10  A solid steel bar of diameter \( d_1 = 25.0 \text{ mm} \) is enclosed by a steel tube of outer diameter \( d_3 = 37.5 \text{ mm} \) and inner diameter \( d_2 = 30.0 \text{ mm} \) (see figure). Both bar and tube are held rigidly by a support at end \( A \) and joined securely to a rigid plate at end \( B \). The composite bar, which has a length \( L = 550 \text{ mm} \), is twisted by a torque \( T = 400 \text{ N} \cdot \text{m} \) acting on the end plate.

(a) Determine the maximum shear stresses \( \tau_1 \) and \( \tau_2 \) in the bar and tube, respectively.

(b) Determine the angle of rotation \( \phi \) (in degrees) of the end plate, assuming that the shear modulus of the steel is \( G = 80 \text{ GPa} \).

(c) Determine the torsional stiffness \( k_T \) of the composite bar.

**Hint:** Use Eqs. 3-44a and b to find the torques in the bar and tube.
Solution 3.8-10  Bar enclosed in a tube

\begin{align*}
d_1 &= 25.0 \text{ mm} \\
d_2 &= 30.0 \text{ mm} \\
d_3 &= 37.5 \text{ mm} \\
G &= 80 \text{ GPa}
\end{align*}

Polar moments of inertia

\begin{align*}
\text{Bar: } I_{p1} &= \frac{\pi}{32} d_1^4 = 38.3495 \times 10^{-9} \text{ m}^4 \\
\text{Tube: } I_{p2} &= \frac{\pi}{32} (d_3^4 - d_2^4) = 114.6229 \times 10^{-9} \text{ m}^4
\end{align*}

Bar: \[ T_1 = T \left( \frac{I_{p1}}{I_{p1} + I_{p2}} \right) = 100.2783 \text{ N} \cdot \text{m} \]
Tube: \[ T_2 = T \left( \frac{I_{p2}}{I_{p1} + I_{p2}} \right) = 299.717 \text{ N} \cdot \text{m} \]

(a) Maximum shear stresses

\begin{align*}
\text{Bar: } \tau_1 &= \frac{T_1 (d_1/2)}{I_{p1}} = 32.7 \text{ MPa} \\
\text{Tube: } \tau_2 &= \frac{T_2 (d_3/2)}{I_{p2}} = 49.0 \text{ MPa}
\end{align*}

(b) Angle of rotation of end plate

\[ \phi = \frac{T_1 \lambda}{G I_{p1}} = \frac{T_2 \lambda}{G I_{p2}} = 0.017977 \text{ rad} \]
\[ \phi = 1.03^\circ \]

(c) Torsional stiffness

\[ k_T = \frac{T}{\phi} = 22.3 \text{ kN} \cdot \text{m} \]

Problem 3.8-11  A solid steel bar of diameter \(d_1 = 1.50 \text{ in.}\) is enclosed by a steel tube of outer diameter \(d_3 = 2.25 \text{ in.}\) and inner diameter \(d_2 = 1.75 \text{ in.}\) (see figure). Both bar and tube are held rigidly by a support at end \(A\) and joined securely to a rigid plate at end \(B\). The composite bar, which has length \(L = 30.0 \text{ in.}\), is twisted by a torque \(T = 5000 \text{ lb-in.}\) acting on the end plate.

(a) Determine the maximum shear stresses \(\tau_1\) and \(\tau_2\) in the bar and tube, respectively.
(b) Determine the angle of rotation \(\phi\) (in degrees) of the end plate, assuming that the shear modulus of the steel is \(G = 11.6 \times 10^6 \text{ psi}\).
(c) Determine the torsional stiffness \(k_T\) of the composite bar. (Hint: Use Eqs. 3-44a and b to find the torques in the bar and tube.)
Section 3.8  Statically Indeterminate Torsional Members

Solution 3.8-11  Bar enclosed in a tube

\[ T = 5000 \text{ lb-in.} \]

\[ L = 30.0 \text{ in.} \]

\[ \begin{align*}
  d_1 &= 1.50 \text{ in.} \\
  d_2 &= 1.75 \text{ in.} \\
  d_3 &= 2.25 \text{ in.} \\
  G &= 11.6 \times 10^6 \text{ psi}
\end{align*} \]

**Polar Moments of Inertia**

Bar: \( I_{P1} = \frac{\pi}{32} d_1^4 = 0.497010 \text{ in.}^4 \)

Tube: \( I_{P2} = \frac{\pi}{32} (d_3^4 - d_2^4) = 1.595340 \text{ in.}^4 \)

\( \text{Torques in the bar (1) and tube (2) from Eqs. (3-44a and b)} \)

\[ \begin{align*}
  \text{Bar: } T_1 &= T \left( \frac{I_{P1}}{I_{P1} + I_{P2}} \right) = 1187.68 \text{ lb-in.} \\
  \text{Tube: } T_2 &= T \left( \frac{I_{P2}}{I_{P1} + I_{P2}} \right) = 3812.32 \text{ lb-in.}
\end{align*} \]

(a) **Maximum Shear Stresses**

Bar: \( \tau_1 = \frac{T_1 (d_1/2)}{I_{P1}} = 1790 \text{ psi} \)

Tube: \( \tau_2 = \frac{T_2 (d_3/2)}{I_{P2}} = 2690 \text{ psi} \)

(b) **Angle of Rotation of End Plate**

\[ \phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.006180015 \text{ rad} \]

\[ \phi = 0.354^\circ \]

(c) **Torsional Stiffness**

\[ k_T = \frac{T}{\phi} = 809 \text{ k-in.} \]

Problem 3.8-12  The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are \( d_1 = 40 \text{ mm} \) for the brass core and \( d_2 = 50 \text{ mm} \) for the steel sleeve. The shear moduli of elasticity are \( G_B = 36 \text{ GPa} \) for the brass and \( G_S = 80 \text{ GPa} \) for the steel.

Assuming that the allowable shear stresses in the brass and steel are \( \tau_B = 48 \text{ MPa} \) and \( \tau_S = 80 \text{ MPa} \), respectively, determine the maximum permissible torque \( T_{\text{max}} \) that may be applied to the shaft. (Hint: Use Eqs. 3-44a and b to find the torques.)

Solution 3.8-12  Composite shaft shrink fit

\[ \begin{align*}
  d_1 &= 40 \text{ mm} \\
  d_2 &= 50 \text{ mm} \\
  G_B &= 36 \text{ GPa} \\
  G_S &= 80 \text{ GPa}
\end{align*} \]

Allowable stresses:

\[ \tau_B = 48 \text{ MPa} \quad \tau_S = 80 \text{ MPa} \]
Problem 3.8-13  The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 1.6$ in. for the brass core and $d_2 = 2.0$ in. for the steel sleeve. The shear moduli of elasticity are $G_b = 5400$ ksi for the brass and $G_s = 12,000$ ksi for the steel.

Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 4500$ psi and $\tau_s = 7500$ psi, respectively, determine the maximum permissible torque $T_{max}$ that may be applied to the shaft. (Hint: Use Eqs. 3-44a and b to find the torques.)
Solution 3.8-13 Composite shaft shrink fit

Steel sleeve

Brass core

\[ d_1 = 1.6 \text{ in.} \]
\[ d_2 = 2.0 \text{ in.} \]
\[ G_B = 5,400 \text{ psi} \quad G_S = 12,000 \text{ psi} \]

Allowable stresses:
\[ \tau_B = 4500 \text{ psi} \quad \tau_S = 7500 \text{ psi} \]

**BRASS CORE (ONLY)**

\[ T_B = \frac{G_B I_{PB}}{G_B I_{PB} + G_S I_{PS}} \]
\[ I_{PB} = \frac{\pi}{32} d_1^4 = 0.643398 \text{ in.}^4 \]
\[ G_B I_{PB} = 3.47435 \times 10^6 \text{ lb-in.}^2 \]

**STEEL SLEEVE (ONLY)**

\[ T_S = \frac{2 \tau_S I_{PS}}{d_2} \]
\[ I_{PS} = \frac{\pi}{32} (d_2^4 - d_1^4) = 0.927398 \text{ in.}^4 \]
\[ G_S I_{PS} = 11.1288 \times 10^6 \text{ lb-in.}^2 \]

**TORQUES**

Total torque: \[ T = T_B + T_S \]
\[ \text{Eq. (3-44 a): } T_B = T \left( \frac{G_B I_{PB}}{G_B I_{PB} + G_S I_{PS}} \right) \]
\[ = 0.237918 T \]
\[ \text{Eq. (3-44 b): } T_S = T \left( \frac{G_S I_{PS}}{G_B I_{PB} + G_S I_{PS}} \right) \]
\[ = 0.762082 T \]
\[ T = T_B + T_S \] (CHECK)

**ALLOWABLE TORQUE**

\[ T_B = 0.237918 T \]
\[ = \frac{2(4500 \text{ psi})(0.643398 \text{ in.}^4)}{1.6 \text{ in.}} \]
\[ T = 15.21 \text{ k-in.} \]

\[ T_S = 0.762082 T \]
\[ = \frac{2(7500 \text{ psi})(0.927398 \text{ in.}^4)}{2.0 \text{ in.}} \]
\[ T = 9.13 \text{ k-in.} \]

Steel sleeve governs \[ T_{\text{max}} = 9.13 \text{ k-in.} \]

---

Problem 3.8-14 A steel shaft \((G_s = 80 \text{ GPa})\) of total length \(L = 3.0 \text{ m}\) is encased for one-third of its length by a brass sleeve \((G_b = 40 \text{ GPa})\) that is securely bonded to the steel (see figure). The outer diameters of the shaft and sleeve are \(d_1 = 70 \text{ mm}\) and \(d_2 = 90 \text{ mm}\) respectively.
(a) Determine the allowable torque $T_1$ that may be applied to the ends of the shaft if the angle of twist between the ends is limited to 8.0°.
(b) Determine the allowable torque $T_2$ if the shear stress in the brass is limited to $\tau_b = 70$ MPa.
(c) Determine the allowable torque $T_3$ if the shear stress in the steel is limited to $\tau_s = 110$ MPa.
(d) What is the maximum allowable torque $T_{max}$ if all three of the preceding conditions must be satisfied?

Solution 3.8-14

(a) **Allowable Torque $T_1$ Based on Twist at Ends of 8 Degrees**
First find torques in steel ($T_s$) & brass ($T_b$) in segment in which they are joined - 1 degree stat-indet; use $T_s$ as the internal redundant; see equ. 3-44a in text example

$$T_s = T_1 \left( \frac{G_s I_p s}{G_s I_p s + G_b I_p b} \right)$$

For middle term, brass sleeve & steel shaft twist the same so could use $T_b = T_1 - T_s$; $T_b = T_1 \left( \frac{G_b I_p b}{G_s I_p s + G_b I_p b} \right)$

now find twist of 3 segments:

$$\phi = \frac{T_1 L}{4 G_b I_p b} + \frac{T_s L}{4 G_s I_p s} + \frac{T_b L}{2 G_s I_p s}$$

Let $\phi_a = \phi_{allow}$; substitute expression for $T_b$ then simplify; finally, solve for $T_1$, allow

$$\phi_a = \frac{T_1 L}{4 G_b I_p b} + \frac{T_1 L}{4 G_s I_p s} \left( \frac{G_s I_p s}{G_s I_p s + G_b I_p b} \right) + \frac{T_1 L}{2 G_s I_p s}$$

$$\phi_a = \frac{T_1 L}{4 G_b I_p b} + \frac{T_1 L}{4 G_s I_p s} \left( \frac{1}{G_s I_p s + G_b I_p b} + \frac{1}{G_s I_p s} + \frac{2}{G_s I_p s} \right)$$

$$T_{1, allow} = \frac{4 \phi_a}{L} \left[ \frac{G_b I_p b (G_s I_p s + G_b I_p b) G_s I_p s}{G_s I_p s^2 + 4 G_b I_p b G_s I_p s + 2 G_b^2 I_p b^2} \right]$$
Numerical values \( \phi_a = 8\left(\frac{\pi}{180}\right) \text{ rad} \)

\( G_s = 80 \text{ GPa} \quad G_b = 40 \text{ GPa} \quad L = 3.0 \text{ m} \)

\( d_1 = 70 \text{ mm} \quad d_2 = 90 \text{ mm} \)

\( I_{Ps} = \frac{\pi}{32} d_1^4 \quad I_{Ps} = 2.357 \times 10^{-6} \text{ m}^4 \)

\( I_{Pb} = \frac{\pi}{32} \left(d_2^4 - d_1^4\right) \quad I_{Pb} = 4.084 \times 10^{-6} \text{ m}^4 \)

\( T_{1, \text{allow}} = 9.51 \text{ kN} \cdot \text{m} \leftarrow \)

(b) Allowable torque \( T_2 \) based on allowable shear stress in brass, \( \tau_b \)

\( \tau_b = 70 \text{ MPa} \)

First check hollow segment 1 (brass sleeve only)

\[ \tau = \frac{T_2}{I_{Pb}} \quad T_{2, \text{allow}} = \frac{2\tau_b I_{Pb}}{d_2} \]

\( T_{2, \text{allow}} = 6.35 \text{ kN} \cdot \text{m} \leftarrow \)

controls over \( T_2 \) below also check segment 2 with brass sleeve over steel shaft

\[ \tau = \frac{T_2}{I_{Pb}} \quad \text{where from stat-indet analysis above} \]

\[ T_b = T_2 \left(\frac{G_b I_{Pb}}{G_s I_{Ps} + G_b I_{Pb}}\right) \]

\[ T_{2, \text{allow}} = \frac{2\tau_b (G_s I_{Ps} + G_b I_{Pb})}{d_2 G_b} \]

\( T_{2, \text{allow}} = 13.69 \text{ kN} \cdot \text{m} \)

so \( T_2 \) for hollow segment controls

(c) Allowable torque \( T_3 \) based on allowable shear stress in steel, \( \tau_s \)

\( \tau_s = 110 \text{ MPa} \)

First check segment 2 with brass sleeve over steel shaft

\[ \tau = \frac{T_3}{I_{Ps}} \quad \text{where from stat-indet analysis above} \]

\[ T_s = T_3 \left(\frac{G_s I_{Ps}}{G_s I_{Ps} + G_b I_{Pb}}\right) \]

\[ T_{3, \text{allow}} = \frac{2\tau_s (G_s I_{Ps} + G_b I_{Pb})}{d_1 G_s} \]

\( T_{3, \text{allow}} = 13.83 \text{ kN} \cdot \text{m} \)

also check segment 3 with steel shaft alone

\[ \tau = \frac{T_3}{I_{Ps}} \quad T_{3, \text{allow}} = \frac{2\tau_s I_{Ps}}{d_1} \]

\( T_{3, \text{allow}} = 7.41 \text{ kN} \cdot \text{m} \leftarrow \) controls over \( T_3 \) above

(d) \( T_{\text{max}} \) if all preceding conditions must be considered

from (b) above

\[ T_{\text{max}} = 6.35 \text{ kN} \cdot \text{m} \leftarrow \) max. shear stress in hollow brass sleeve in segment 1 controls overall
Problem 3.8-15  A uniformly tapered aluminum-alloy tube $AB$ of circular cross section and length $L$ is fixed against rotation at $A$ and $B$, as shown in the figure. The outside diameters at the ends are $d_A$ and $d_B = 2d_A$. A hollow section of length $L/2$ and constant thickness $t = d_A/10$ is cast into the tube and extends from $B$ halfway toward $A$. Torque $T_0$ is applied at $L/2$.

(a) Find the reactive torques at the supports, $T_A$ and $T_B$. Use numerical values as follows: $d_A = 2.5$ in., $L = 48$ in., $G = 3.9 \times 10^6$ psi, $T_0 = 40,000$ in.-lb.

(b) Repeat (a) if the hollow section has constant diameter $d_A$.

Solution 3.8-15

Solution approach-superposition: select $T_B$ as the redundant ($1^\circ$ SI)

\[
\phi_1 = \frac{608T_0L}{81G\pi d_A^4} \\
\phi_2 = \frac{T_0}{Gd_A^4}
\]

See Prob. 3.4-13 for results for $\phi_2$ for Parts $a$ & $b$

\[
\phi_{2a} = 3.868 \frac{T_0L}{Gd_A^4} \\
\phi_{2b} = 3.057 \frac{T_0L}{Gd_A^4}
\]
(a) **Reactive torques, \( T_A \) & \( T_B \), for case of constant thickness of hollow section of tube**

Compatibility equation: \( \phi_1 - \phi_2 = 0 \)

\( T_B = \) redundant
\( T_0 = 40000 \text{ in.-lb} \)

\[ T_B = \left( \frac{608T_0L}{81G \pi d_A^4} \right) \left( \frac{Gd_A^4}{3.86804L} \right) \]

\[ T_B = 1.94056 \frac{T_0}{\pi} \quad T_B = 24708 \text{ in-lb} \leftarrow \]

\[ T_A = T_0 - T_B \quad T_A = 15292 \text{ in-lb} \leftarrow \]

\[ T_A + T_B = 40,000 \text{ in.-lb (check)} \]

---

**Problem 3.8-16** A hollow circular tube \( A \) (outer diameter \( d_A \), wall thickness \( t_A \)) fits over the end of a circular tube \( B \) (\( d_B \), \( t_B \)), as shown in the figure. The far ends of both tubes are fixed. Initially, a hole through tube \( B \) makes an angle \( \beta \) with a line through two holes in tube \( A \). Then tube \( B \) is twisted until the holes are aligned, and a pin (diameter \( d_p \)) is placed through the holes. When tube \( B \) is released, the system returns to equilibrium. Assume that \( G \) is constant.

(a) Use superposition to find the reactive torques \( T_A \) and \( T_B \) at the supports.

(b) Find an expression for the maximum value of \( \beta \) if the shear stress in the pin, \( \tau_p \), cannot exceed \( \tau_p, \text{allow} \).

(c) Find an expression for the maximum value of \( \beta \) if the shear stress in the tubes, \( \tau_c \), cannot exceed \( \tau_c, \text{allow} \).

(d) Find an expression for the maximum value of \( \beta \) if the bearing stress in the pin at \( C \) cannot exceed \( \sigma_b, \text{allow} \).

---

**Solution 3.8-16**

(a) **Superposition to find torque reactions - use \( T_B \) as the redundant**

Compatibility: \( \phi_{B1} + \phi_{B2} = 0 \)

\( \phi_{B1} = -\beta \) < joint tubes by pin then release end \( B \)

\[ \phi_{B2} = T_{BL} \frac{1}{G} \left( \frac{1}{I_{PA}} + \frac{1}{I_{PB}} \right) \]

\[ \phi_{B2} = T_{BL} \frac{I_{PB} + I_{PA}}{G I_{PA} I_{PB}} \]

\[ T_B = \frac{G \beta}{L} \left( \frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right) \leftarrow \]

\[ T_A = -T_B \leftarrow \text{statics} \]

---

(b) **Allowable shear in pin restricts magnitude of \( \beta \)**

Torsion \( T_B = \) force couple \( Vd_B \) with \( V = \) shear in pin at \( C \)

\[ V = \frac{T_B}{d_B} \quad \tau_p = \frac{V}{A_s} \]

\[ \tau_p, \text{allow} = \frac{G \beta}{d_B} \left( \frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right) \]

\[ \beta_{\text{max}} = \tau_p, \text{allow} \left( \frac{L}{4G} \right) \left[ \left( \frac{I_{PB} + I_{PA}}{I_{PA} I_{PB}} \right) d_B \pi d_p^2 \right] \leftarrow \]
(c) **Allowable shear in tubes restricts magnitude of $\beta$**

$$\tau_{\text{max}} = \frac{T_B}{2 \left( I_{PA} + I_{PB} \right)} \frac{d_A}{I_{PA}}$$

or

$$\tau_{\text{max}} = \frac{T_B}{2 \left( I_{PA} + I_{PB} \right)} \frac{d_B}{I_{PB}}$$

simplifying these two equ., then solving for $\beta$ gives:

$$\tau_{\text{max}} = \frac{GB}{L} \left( I_{PA} + I_{PB} \right) \frac{d_A}{2}$$

or

$$\tau_{\text{max}} = \frac{GB}{L} \left( I_{PA} + I_{PB} \right) \frac{d_B}{2}$$

$$\beta_{\text{max}} = \tau_{\text{max}} \left( \frac{2L}{GB} \right) \left( I_{PA} + I_{PB} \right) \left( \frac{d_A}{I_{PA}} \right) \left( \frac{d_B}{I_{PB}} \right)$$

where lesser value of $\beta$ controls

(d) **Allowable bearing stress in pin restricts magnitude of $\beta$**

Torque $T_B$ = force couple $F_B(d_B - t_B)$ or $F_A(d_A - t_A)$, with $F = \text{ave. bearing force on pin at } C$

Bearing stresses from tubes $A$ & $B$ are:

$$\sigma_{ha} = \frac{F_A}{d_P A} \quad \sigma_{hb} = \frac{F_B}{d_P B}$$

$$\sigma_{ha} = \frac{d_A - t_A}{d_P A} \quad \sigma_{hb} = \frac{d_B - t_B}{d_P B}$$

substitute $T_B$ expression from part (a), then simplify $e$ solve for $\beta$

$$\beta_{ha} = \frac{GB}{L} \left( I_{PA} + I_{PB} \right) \frac{d_A}{d_P A}$$

$$\beta_{hb} = \frac{GB}{L} \left( I_{PA} + I_{PB} \right) \frac{d_B}{d_P B}$$

$$\beta_{ha} = \beta \frac{G I_{PA} I_{PB}}{L (I_{PB} + I_{PA})(d_A - t_A)d_P A}$$

$$\beta_{hb} = \beta \frac{G I_{PA} I_{PB}}{L (I_{PB} + I_{PA})(d_B - t_B)d_P B}$$

$$\beta_{max} = \sigma_{b, allow} \frac{L}{G}$$

$$\beta_{max} = \sigma_{b, allow} \frac{L}{G} \left[ \frac{(I_{PB} + I_{PA})(d_A - t_A)d_P A}{I_{PA} I_{PB}} \right]$$

$$\beta_{max} = \sigma_{b, allow} \frac{L}{G} \left[ \frac{(I_{PB} + I_{PA})(d_B - t_B)d_P B}{I_{PA} I_{PB}} \right]$$

where lesser value controls
Strain Energy in Torsion

Problem 3.9-1  A solid circular bar of steel \((G = 11.4 \times 10^6 \text{ psi})\) with length \(L = 30 \text{ in.}\) and diameter \(d = 1.75 \text{ in.}\) is subjected to pure torsion by torques \(T\) acting at the ends (see figure).

(a) Calculate the amount of strain energy \(U\) stored in the bar when the maximum shear stress is 4500 psi.
(b) From the strain energy, calculate the angle of twist \(\phi\) (in degrees).

Solution 3.9-1  Steel bar

\(G = 11.4 \times 10^6 \text{ psi}\)
\(L = 30 \text{ in.}\)
\(d = 1.75 \text{ in.}\)
\(\tau_{\text{max}} = 4500 \text{ psi}\)

\(\tau_{\text{max}} = \frac{16 T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{\text{max}}}{16}\)  \hspace{1cm} (Eq. 1)

\(I_P = \frac{\pi d^4}{32}\)  \hspace{1cm} (Eq. 1)

(a) STRAIN ENERGY

\[ U = \frac{T^2 L}{2 G I_P} = \left(\frac{\pi d^3 \tau_{\text{max}}}{16}\right)^2 \left(\frac{L}{2G}\right) \left(\frac{32}{\pi d^4}\right) \]

(b) ANGLE OF TWIST

\[ U = \frac{T \phi}{2} \quad \phi = \frac{2U}{T} \]

Substitute numerical values:
\[ U = 32.0 \text{ in.-lb} \quad \phi = 0.013534 \text{ rad} = 0.775^\circ\]

Problem 3.9-2  A solid circular bar of copper \((G = 45 \text{ GPa})\) with length \(L = 0.75 \text{ m}\) and diameter \(d = 40 \text{ mm}\) is subjected to pure torsion by torques \(T\) acting at the ends (see figure).

(a) Calculate the amount of strain energy \(U\) stored in the bar when the maximum shear stress is 32 MPa.
(b) From the strain energy, calculate the angle of twist \(\phi\) (in degrees)
Problem 3.9-3  A stepped shaft of solid circular cross sections (see figure) has length \( L = 45 \) in., diameter \( d_2 = 1.2 \) in., and diameter \( d_1 = 1.0 \) in. The material is brass with \( G = 5.6 \times 10^6 \) psi. Determine the strain energy \( U \) of the shaft if the angle of twist is \( 3.0^\circ \).

Solution 3.9-3  Stepped shaft

\[ U = \sum \frac{T^2L}{2GI_p} = \frac{16T^2(L/2)}{\pi Gd_2^4} + \frac{16T^2(L/2)}{\pi Gd_1^4} \]  

(Eq. 1)

Also, \( U = \frac{T \phi}{2} \)  

(Eq. 2)

Equate \( U \) from Eqs. (1) and (2) and solve for \( T \):

\[ T = \frac{\pi Gd_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)} \]

\[ U = \frac{T \phi}{2} = \frac{\pi Gd_1^4 d_2^4 \phi}{32L(d_1^4 + d_2^4)} \quad \phi = \text{radians} \]

Substitute numerical values:

\( U = 22.6 \text{ in.-lb} \)
Problem 3.9-4  A stepped shaft of solid circular cross sections (see figure) has length $L = 0.80\,\text{m}$, diameter $d_2 = 40\,\text{mm}$, and diameter $d_1 = 30\,\text{mm}$. The material is steel with $G = 80\,\text{GPa}$.

Determine the strain energy $U$ of the shaft if the angle of twist is $1.0^\circ$.

Solution 3.9-4  Stepped shaft

Also, $U = \frac{T\phi}{2}$  
(Eq. 2)

Equate $U$ from Eqs. (1) and (2) and solve for $T$:

$$T = \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L} \left( \frac{d_1^4 + d_2^4}{d_1^4 + d_2^4} \right) \phi = \text{radians}$$

Substitute numerical values:

$U = 1.84\,\text{J}$  

Problem 3.9-5  A cantilever bar of circular cross section and length $L$ is fixed at one end and free at the other (see figure). The bar is loaded by a torque $T$ at the free end and by a distributed torque of constant intensity $t$ per unit distance along the length of the bar.

(a) What is the strain energy $U_1$ of the bar when the load $T$ acts alone?
(b) What is the strain energy $U_2$ when the load $t$ acts alone?
(c) What is the strain energy $U_3$ when both loads act simultaneously?

Solution 3.9-5  Cantilever bar with distributed torque

$G = \text{shear modulus}$

$I_p = \text{polar moment of inertia}$

$T = \text{torque acting at free end}$

$t = \text{torque per unit distance}$
Chapter 3 Torsion

Problem 3.9-6 Obtain a formula for the strain energy $U$ of the statically indeterminate circular bar shown in the figure. The bar has fixed supports at ends $A$ and $B$ and is loaded by torques $2T_0$ and $T_0$ at points $C$ and $D$, respectively.

Hint: Use Eqs. 3-46a and b of Example 3-9, Section 3.8, to obtain the reactive torques.

(a) Load $T$ acts alone (Eq. 3-51a)

$$U_1 = \frac{T^2L}{2GI_p} \quad \leftarrow$$

(b) Load $t$ acts alone

From Eq. (3-56) of Example 3-11:

$$U_2 = \frac{T^3L^3}{6GI_p} \quad \leftarrow$$

(c) Both loads act simultaneously

At distance $x$ from the free end:

$$U_3 = \int_0^L (T(x))^2 \, dx = \frac{1}{2GI_p} \int_0^L (T + tx)^2 \, dx$$

$$= \frac{T^2L}{2GI_p} + \frac{t^2L^2}{2GI_p} + \frac{T^3L^3}{6GI_p} \quad \leftarrow$$

NOTE: $U_3$ is not the sum of $U_1$ and $U_2$.

Solution 3.9-6 Statically indeterminate bar

Reactive torques

From Eq. (3-46a):

$$T_A = \frac{(2T_0)\left(\frac{3L}{4}\right)}{L} + \frac{T_0\left(\frac{L}{4}\right)}{L} = \frac{7T_0}{4}$$

$$T_B = 3T_0 - T_A = \frac{5T_0}{4}$$

Internal torques

$$T_{AC} = -\frac{7T_0}{4} \quad T_{CD} = \frac{T_0}{4} \quad T_{DB} = \frac{5T_0}{4}$$

Strain energy (from Eq. 3-53)

$$U = \sum_{i=1}^n \frac{T_i^2L_i}{2GI_{pi}}$$

$$= \frac{1}{2GI_p} \left[ T_{AC}^2 \left(\frac{L}{4}\right) + T_{CD}^2 \left(\frac{L}{2}\right) + T_{DB}^2 \left(\frac{L}{4}\right) \right]$$

$$= \frac{1}{2GI_p} \left[ \left(\frac{7T_0}{4}\right)^2 \left(\frac{L}{4}\right) + \left(\frac{5T_0}{4}\right)^2 \left(\frac{L}{4}\right) \right]$$

$$U = \frac{19T_0^2L}{32GI_p} \quad \leftarrow$$
Problem 3.9-7  A statically indeterminate stepped shaft $ACB$ is fixed at ends $A$ and $B$ and loaded by a torque $T_0$ at point $C$ (see figure). The two segments of the bar are made of the same material, have lengths $L_A$ and $L_B$, and have polar moments of inertia $I_{PA}$ and $I_{PB}$.

Determine the angle of rotation $\phi$ of the cross section at $C$ by using strain energy.

Hint: Use Eq. 3-51b to determine the strain energy $U$ in terms of the angle $\phi$. Then equate the strain energy to the work done by the torque $T_0$. Compare your result with Eq. 3-48 of Example 3-9, Section 3.8.

Solution 3.9-7  Statically indeterminate bar

\[ U = \sum_{i=1}^{n} \frac{GIP_i \phi_i^2}{2L_i} = \frac{GL_P A \phi^2}{2L_A} + \frac{GL_P B \phi^2}{2L_B} \]

\[ = \frac{G\phi^2}{2} \left( \frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B} \right) \]

Work done by the torque $T_0$

\[ W = \frac{T_0 \phi}{2} \]

Equate $U$ and $W$ and solve for $\phi$

\[ \frac{G\phi^2}{2} \left( \frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B} \right) = \frac{T_0 \phi}{2} \]

\[ \phi = \frac{T_0 L_A L_B}{G (I_P L_A + I_P L_B)} \]

(This result agrees with Eq. (3-48) of Example 3-9, Section 3.8.)

Problem 3.9-8  Derive a formula for the strain energy $U$ of the cantilever bar shown in the figure.

The bar has circular cross sections and length $L$. It is subjected to a distributed torque of intensity $t$ per unit distance. The intensity varies linearly from $t = 0$ at the free end to a maximum value $t = t_0$ at the support.
Solution 3.9-8  Cantilever bar with distributed torque

\[ t(x) = t_0 \left( \frac{x}{L} \right) \]

\[ x = \text{distance from right-hand end of the bar} \]

**ELEMENT \( d\xi \)**

Consider a differential element \( d\xi \) at distance \( \xi \) from the right-hand end.

\[ dT = \text{external torque acting on this element} \]

\[ dT = t(\xi) d\xi \]

\[ = t_0 \left( \frac{\xi}{L} \right) d\xi \]

**ELEMENT \( dx \) AT DISTANCE \( x \)**

\[ T(x) = \text{internal torque acting on this element} \]

\[ T(x) = \text{total torque from } x = 0 \text{ to } x = x \]

\[ T(x) = \int_0^x dT = \int_0^x t_0 \left( \frac{\xi}{L} \right) d\xi \]

\[ = \frac{t_0 x^2}{2L} \]

**STRAIN ENERGY OF ELEMENT \( dx \)**

\[ dU = \frac{[T(x)]^2 dx}{2GIp} = \frac{1}{2GIp} \left( \frac{t_0}{2L} \right)^2 x^4 dx \]

\[ = \frac{t_0^2}{8L^2 GIp} x^4 dx \]

**STRAIN ENERGY OF ENTIRE BAR**

\[ U = \int_0^L dU = \frac{t_0^2}{8L^2 GIp} \int_0^L x^4 dx \]

\[ = \frac{t_0^2}{8L^2 GIp} \left( \frac{L^5}{5} \right) \]

\[ U = \frac{t_0^2 L^3}{40GIp} \]
Problem 3.9-9  A thin-walled hollow tube $AB$ of conical shape has constant thickness $t$ and average diameters $d_A$ and $d_B$ at the ends (see figure).

(a) Determine the strain energy $U$ of the tube when it is subjected to pure torsion by torques $T$.

(b) Determine the angle of twist $\phi$ of the tube.

Note: Use the approximate formula $I_P \approx \pi d^3 t/4$ for a thin circular ring; see Case 22 of Appendix D.

Solution 3.9-9  Thin-walled, hollow tube

\[ t = \text{thickness} \]
\[ d_A = \text{average diameter at end } A \]
\[ d_B = \text{average diameter at end } B \]
\[ d(x) = \text{average diameter at distance } x \text{ from end } A \]
\[ d(x) = d_A + \left( \frac{d_B - d_A}{L} \right) x \]

Polar moment of inertia
\[ I_P = \frac{\pi d^3 t}{4} \]
\[ I_P(x) = \frac{\pi [d(x)]^3 t}{4} = \frac{\pi t}{4} \left[ d_A + \left( \frac{d_B - d_A}{L} \right) x \right]^3 \]

(a) Strain energy (from Eq. 3-54)
\[ U = \int_0^L \frac{T^2 dx}{2G I_P(x)} = \frac{2T^2}{\pi G t} \int_0^L \frac{dx}{\left[ d_A + \left( \frac{d_B - d_A}{L} \right) x \right]^3} \]

(Eq. 1)

From Appendix C:
\[ \int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2} \]

Therefore,
\[ \int_0^L \frac{dx}{d_A + \left( \frac{d_B - d_A}{L} \right) x} = \frac{1}{\frac{2(d_B - d_A)}{L}} \left[ \frac{d_B - d_A}{L} \right]^2 \]

\[ = -\frac{L}{2(d_B - d_A)(d_B)^2} + \frac{L}{2(d_B - d_A)(d_B)^2} \]

\[ = \frac{L(d_A + d_B)}{2d_A^2 d_B^2} \]

Substitute this expression for the integral into the equation for $U$ (Eq. 1):
\[ U = \frac{2T^2}{\pi G t} \frac{L(d_A + d_B)}{2d_A^2 d_B^2} = \frac{T^2 L}{\pi G t} \left( \frac{d_A + d_B}{d_A^2 d_B^2} \right) \]

(b) Angle of twist

Work of the torque $T$: $W = \frac{T \phi}{2}$
\[ W = U \]
\[ \frac{T \phi}{2} = \frac{T^2 L(d_A + d_B)}{\pi G t d_A^2 d_B^2} \]

Solve for $\phi$:
\[ \phi = \frac{2T L(d_A + d_B)}{\pi G t d_A^2 d_B^2} \]
Problem 3.9-10  A hollow circular tube $A$ fits over the end of a solid circular bar $B$, as shown in the figure. The far ends of both bars are fixed. Initially, a hole through bar $B$ makes an angle $\beta$ with a line through two holes in tube $A$. Then bar $B$ is twisted until the holes are aligned, and a pin is placed through the holes. When bar $B$ is released and the system returns to equilibrium, what is the total strain energy $U$ of the two bars? (Let $I_{PA}$ and $I_{PB}$ represent the polar moments of inertia of bars $A$ and $B$, respectively. The length $L$ and shear modulus of elasticity $G$ are the same for both bars.)

Solution 3.9-10  Circular tube and bar

**Tube $A$**

$T = \text{torque acting on the tube}$

$\phi_A = \text{angle of twist}$

**Bar $B$**

$T = \text{torque acting on the bar}$

$\phi_B = \text{angle of twist}$

**Compatibility**

$\phi_A + \phi_B = \beta$

**Force-displacement relations**

$\phi_A = \frac{TL}{GI_{PA}}$  $\phi_B = \frac{TL}{GI_{PB}}$

Substitute into the equation of compatibility and solve for $T$:

$T = \frac{\beta G}{L} \left( \frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right)$

**Strain energy**

$U = \sum \frac{T^2 L}{2GI_p} = \frac{T^2 L}{2GI_{PA}} + \frac{T^2 L}{2GI_{PB}}$

$= \frac{T^2 L}{2G} \left( \frac{1}{I_{PA}} + \frac{1}{I_{PB}} \right)$

Substitute for $T$ and simplify:

$U = \frac{\beta^2 G}{2L} \left( \frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right)$
Problem 3.9-11 A heavy flywheel rotating at $n$ revolutions per minute is rigidly attached to the end of a shaft of diameter $d$ (see figure). If the bearing at $A$ suddenly freezes, what will be the maximum angle of twist $\phi$ of the shaft? What is the corresponding maximum shear stress in the shaft? (Let $L = \text{length of the shaft}$, $G = \text{shear modulus of elasticity}$, and $I_m = \text{mass moment of inertia of the flywheel about the axis of the shaft}$. Also, disregard friction in the bearings at $B$ and $C$ and disregard the mass of the shaft.)

*Hint*: Equate the kinetic energy of the rotating flywheel to the strain energy of the shaft.

Solution 3.9-11 Rotating flywheel

---

**Kinetic energy of flywheel**

\[ \text{K.E.} = \frac{1}{2} I_m v^2 \]

\[ v = \frac{2\pi n}{60} \]

\[ n = \text{rpm} \]

\[ \text{K.E.} = \frac{1}{2} I_m \left( \frac{2\pi n}{60} \right)^2 \]

\[ = \frac{\pi^2 n^2 I_m}{1800} \]

**Units:**

$I_m = \text{(force)(length)(second)}^2$

$\omega = \text{radians per second}$

$\text{K.E.} = \text{(length)(force)}$

**Strain energy of shaft (from Eq. 3-51b)**

\[ U = \frac{G I_m \phi^2}{2L} \]

\[ I_m = \frac{\pi d^4}{32} \]

\[ d = \text{diameter of shaft} \]

\[ U = \frac{\pi G d^4 \phi^2}{64L} \]

**Units:**

$G = \frac{\text{(force)}}{\text{(length)}^2}$

$I_P = \text{(length)}^4$

$\phi = \text{radians}$

$L = \text{length}$

$U = \text{(length)(force)}$

**Equate kinetic energy and strain energy**

\[ \text{K.E.} = U \]

\[ \frac{\pi^2 n^2 I_m}{1800} = \frac{\pi G d^4 \phi^2}{64L} \]

Solve for $\phi$:

\[ \phi = \frac{2n}{15d^2} \sqrt{\frac{2\pi I_m L}{G}} \]

**Maximum shear stress**

\[ \tau = \frac{T(d/2)}{I_p} \]

\[ \phi = \frac{TL}{G I_p} \]

Eliminate $T$:

\[ \tau = \frac{Gd\phi}{2L} \]

\[ \tau_{\max} = \frac{Gd2n}{2L15d^2} \sqrt{\frac{2\pi I_m L}{G}} \]

\[ \tau_{\max} = \frac{n}{15d} \frac{\sqrt{2\pi G I_m \phi}}{L} \]
CHAPTER 3  Torsion

Thin-Walled Tubes

Problem 3.10-1  A hollow circular tube having an inside diameter of 10.0 in. and a wall thickness of 1.0 in. (see figure) is subjected to a torque $T = 1200$ k-in. Determine the maximum shear stress in the tube using (a) the approximate theory of thin-walled tubes, and (b) the exact torsion theory. Does the approximate theory give conservative or nonconservative results?

Solution 3.10-1  Hollow circular tube

$T = 1200$ k-in.
$t = 1.0$ in.
$r = \text{radius to median line}$
$r = 5.5$ in.
$d_2 = \text{outside diameter} = 12.0$ in.
$d_1 = \text{inside diameter} = 10.0$ in.

Approximate theory (Eq. 3-63)

$\tau_1 = \frac{T}{2\pi t^2} = \frac{1200 \text{ k-in.}}{2\pi (5.5 \text{ in.})^2 (1.0 \text{ in.})} = 6314 \text{ psi}$

$\tau_{\text{approx}} = 6310 \text{ psi}$

Exact theory (Eq. 3-11)

$\tau_2 = \frac{T(d_2/2)}{I_p} = \frac{Td_2}{2 \left( \frac{\pi}{32} \right) (d_2^4 - d_1^4)}$

$= \frac{16(1200 \text{ k-in.})(12.0 \text{ in.})}{\pi[(12.0 \text{ in.})^4 - (10.0 \text{ in.})^4]}$

$= 6831 \text{ psi}$

$\tau_{\text{exact}} = 6830 \text{ psi}$

Because the approximate theory gives stresses that are too low, it is nonconservative. Therefore, the approximate theory should only be used for very thin tubes.

Problem 3.10-2  A solid circular bar having diameter $d$ is to be replaced by a rectangular tube having cross-sectional dimensions $d \times 2d$ to the median line of the cross section (see figure). Determine the required thickness $t_{\text{min}}$ of the tube so that the maximum shear stress in the tube will not exceed the maximum shear stress in the solid bar.
### Solution 3.10-2 Bar and tube

**Solid bar**

\[ \tau_{\text{max}} = \frac{16T}{\pi d^3} \quad \text{(Eq. 3-12)} \]

**Rectangular tube**

\[ \tau_{\text{max}} = \frac{T}{2tA_m} = \frac{T}{4td^2} \quad \text{(Eq. 3-61)} \]

Equate the maximum shear stresses and solve for \( t \)

\[ \frac{16T}{\pi d^3} = \frac{T}{4td^2} \]

\[ t_{\text{min}} = \frac{\pi d}{64} \quad \leftarrow \]

If \( t > t_{\text{min}} \), the shear stress in the tube is less than the shear stress in the bar.

### Problem 3.10-3

A thin-walled aluminum tube of rectangular cross section (see figure) has a centerline dimensions \( b = 6.0 \) in. and \( h = 4.0 \) in. The wall thickness \( t \) is constant and equal to 0.25 in.

(a) Determine the shear stress in the tube due to a torque \( T = 15 \) k-in.

(b) Determine the angle of twist (in degrees) if the length \( L \) of the tube is 50 in. and the shear modulus \( G \) is \( 4.0 \times 10^6 \) psi.

### Solution 3.10-3 Thin-walled tube

**Eq. (3-64):** \( A_m = bh = 24.0 \) in.\(^2\)

**Eq. (3-71) with \( t_1 = t_2 = t \):**

\[ J = \frac{2b^2h^2t}{b + h} = 28.8 \text{ in.}^4 \]

(a) **Shear stress (Eq. 3-61)**

\[ \tau = \frac{T}{2tA_m} = 1250 \text{ psi} \quad \leftarrow \]

(b) **Angle of twist (Eq. 3-72)**

\[ \phi = \frac{TL}{GJ} = 0.0065104 \text{ rad} \]

\[ = 0.373^\circ \quad \leftarrow \]

---

- Solution 3.10-2 Bar and tube
- Solution 3.10-3 Thin-walled tube
Problem 3.10-4  A thin-walled steel tube of rectangular cross section (see figure) has centerline dimensions \( b = 150 \text{ mm} \) and \( h = 100 \text{ mm} \). The wall thickness \( t \) is constant and equal to 6.0 mm.

(a) Determine the shear stress in the tube due to a torque \( T = 1650 \text{ N} \cdot \text{m} \).

(b) Determine the angle of twist (in degrees) if the length \( L \) of the tube is 1.2 m and the shear modulus \( G \) is 75 GPa.

\[
\begin{align*}
\text{Solution 3.10-4 Thin-walled tube} \\
\quad & b = 150 \text{ mm} \\
\quad & h = 100 \text{ mm} \\
\quad & t = 6.0 \text{ mm} \\
\quad & T = 1650 \text{ N} \cdot \text{m} \\
\quad & L = 1.2 \text{ m} \\
\quad & G = 75 \text{ GPa} \\
\end{align*}
\]

Eq. (3-64): \( A_m = bh = 0.015 \text{ m}^2 \)

Eq. (3-71) with \( t_1 = t_2 = t \):

\[
J = \frac{2b^2h^2t}{b + h}
\]

\[ J = 10.8 \times 10^{-6} \text{ m}^4 \]

(a) Shear stress (Eq. 3-61)

\[
\tau = \frac{T}{2tA_m} = 9.17 \text{ MPa}
\]

(b) Angle of twist (Eq. 3-72)

\[
\phi = \frac{TL}{GJ} = 0.002444 \text{ rad}
\]

\[
\phi = 0.140^\circ
\]

Problem 3.10-5  A thin-walled circular tube and a solid circular bar of the same material (see figure) are subjected to torsion. The tube and bar have the same cross-sectional area and the same length.

What is the ratio of the strain energy \( U_1 \) in the tube to the strain energy \( U_2 \) in the solid bar if the maximum shear stresses are the same in both cases? (For the tube, use the approximate theory for thin-walled bars.)

\[
\begin{align*}
\text{Solution 3.10-5 Thin-walled tube (1)} \\
A_m &= \pi r^2 \\
J &= 2\pi r^4 t \\
A &= 2\pi rt \\
\tau_{\text{max}} &= \frac{T}{2tA_m} = \frac{T}{2\pi r^4 t} \\
T &= 2\pi r^4 t \tau_{\text{max}} \\
U_1 &= \frac{T^2L}{2GJ} = \frac{(2\pi r^2 t \tau_{\text{max}})^2 L}{2G(2\pi r^4 t)} \\
&= \frac{\pi r^2 t \tau_{\text{max}}^2 L}{G} \\
\text{But } rt &= \frac{A}{2\pi} \\
\therefore \quad U_1 &= \frac{Ar_{\text{max}}^2 L}{2G}
\end{align*}
\]

\[
\begin{align*}
\text{Solid bar (2)} \\
A &= \pi r_2^2 \\
I_p &= \frac{\pi}{2} r_2^4 \\
\tau_{\text{max}} &= \frac{T r_2}{I_p} = \frac{2T}{\pi r_2^2} = T \frac{\pi r_2^3 \tau_{\text{max}}^2}{2} \\
U_2 &= \frac{T^2L}{2GI_p} = \frac{\pi r_2^2 \tau_{\text{max}}^2 L}{8G \left( \frac{\pi}{2} r_2^4 \right)} = \frac{\pi r_2^2 \tau_{\text{max}}^2 L}{4G} \\
\text{But } \pi r_2^2 &= A \\
\therefore \quad U_2 &= A r_{\text{max}}^2 L \quad \frac{\pi}{4G}
\end{align*}
\]

\[
\frac{U_1}{U_2} = 2
\]
Problem 3.10-6  Calculate the shear stress \( \tau \) and the angle of twist \( \phi \) (in degrees) for a steel tube \( (G = 76 \text{ GPa}) \) having the cross section shown in the figure. The tube has length \( L = 1.5 \text{ m} \) and is subjected to a torque \( T = 10 \text{ kN} \cdot \text{m} \).

**Solution 3.10-6  Steel tube**

\[
A_m = \pi r^2 + 2br
\]
\[
A_m = \pi (50 \text{ mm})^2 + 2(100 \text{ mm})(50 \text{ mm})
= 17,850 \text{ mm}^2
\]
\[
L_m = 2b + 2\pi r
= 2(100 \text{ mm}) + 2\pi(50 \text{ mm})
= 514.2 \text{ mm}
\]
\[
J = \frac{4tA_m^2}{L_m} = \frac{4(8 \text{ mm})(17,850 \text{ mm}^2)^2}{514.2 \text{ mm}}
= 19.83 \times 10^6 \text{ mm}^4
\]

\[
\tau = \frac{T}{2tA_m} = \frac{10 \text{ kN} \cdot \text{m}}{2(8 \text{ mm})(17,850 \text{ mm}^2)}
= 35.0 \text{ MPa}
\]

\[
\phi = \frac{TL}{GJ} = \frac{(10 \text{ kN} \cdot \text{m})(1.5 \text{ m})}{(76 \text{ GPa})(19.83 \times 10^6 \text{ mm}^4)}
= 0.00995 \text{ rad}
= 0.570^\circ
\]

Problem 3.10-7  A thin-walled steel tube having an elliptical cross section with constant thickness \( t \) (see figure) is subjected to a torque \( T = 18 \text{ k-in} \).

Determine the shear stress \( \tau \) and the rate of twist \( \theta \) (in degrees per inch) if \( G = 12 \times 10^6 \text{ psi}, t = 0.2 \text{ in.}, a = 3 \text{ in.}, \) and \( b = 2 \text{ in.} \)
(Note: See Appendix D, Case 16, for the properties of an ellipse.)
CHAPTER 3  Torsion

Problem 3.10-8  A torque \(T\) is applied to a thin-walled tube having a cross section in the shape of a regular hexagon with constant wall thickness \(t\) and side length \(b\) (see figure).

Obtain formulas for the shear stress \(\tau\) and the rate of twist \(\theta\).

Solution 3.10-8  Regular hexagon

\[
\begin{align*}
\beta &= 60^\circ \quad n = 6 \\
A_m &= \frac{nb^2}{4} \cot \frac{\beta}{2} = \frac{6b^2}{4} \cot 30^\circ \\
&= \frac{3\sqrt{3}b^2}{2} \\
L_m &= 6b
\end{align*}
\]

Solution 3.10-7  Elliptical tube

\[
T = 18 \text{ k-in.} \\
G = 12 \times 10^6 \text{ psi} \\
t = \text{constant} \\
t = 0.2 \text{ in.} \quad a = 3.0 \text{ in.} \quad b = 2.0 \text{ in.}
\]

Shear stress

\[
\tau = \frac{T}{2ta_m} = \frac{18 \text{ k-in.}}{2(0.2 \text{ in.})(18.850 \text{ in.}^2)} = 2390 \text{ psi}
\]

Angle of twist per unit length (rate of twist)

\[
\theta = \frac{\phi}{L} = \frac{T}{GJ} = \frac{18 \text{ k-in.}}{(12 \times 10^6 \text{ psi})(17.92 \text{ in.})^3} = 83.73 \times 10^{-6} \text{ rad/in.} = 0.0048^\circ/\text{in.}
\]

From Appendix D, Case 16:

\[
\begin{align*}
A_m &= \pi ab = \pi(3.0 \text{ in.})(2.0 \text{ in.}) = 18.850 \text{ in.}^2 \\
L_m &\approx \pi[1.5(a + b) - \sqrt{ab}] \\
&= \pi[1.5(5.0 \text{ in.}) - \sqrt{6.0 \text{ in.}^2}] = 15.867 \text{ in.} \\
J &= \frac{4ta_m^2}{L_m} = \frac{4(0.2 \text{ in.})(18.850 \text{ in.}^2)^2}{15.867 \text{ in.}} \\
&= 17.92 \text{ in.}^4
\end{align*}
\]
**Problem 3.10-9** Compare the angle of twist \( \phi_1 \) for a thin-walled circular tube (see figure) calculated from the approximate theory for thin-walled bars with the angle of twist \( \phi_2 \) calculated from the exact theory of torsion for circular bars.

(a) Express the ratio \( \phi_1/\phi_2 \) in terms of the nondimensional ratio \( \beta = r/t \).

(b) Calculate the ratio of angles of twist for \( \beta = 5, 10, \) and \( 20 \). What conclusion about the accuracy of the approximate theory do you draw from these results?

**Solution 3.10-9 Thin-walled tube**

**Approximate theory**

\[
\phi_1 = \frac{TL}{GJ} J = 2\pi r^3 t \quad \phi_1 = \frac{TL}{2\pi Gr^3 t}
\]

**Exact theory**

\[
\phi_2 = \frac{TL}{GI_p} \quad \text{From Eq. (3-17): } I_p = \frac{\pi t}{2}(4r^2 + t^2)
\]

\[
\phi_2 = \frac{TL}{GI_p} = \frac{2TL}{\pi Gr(4r^2 + t^2)}
\]

(a) **Ratio**

\[
\frac{\phi_1}{\phi_2} = \frac{4r^2 + t^2}{4r^2} = 1 + \frac{t^2}{4r^2}
\]

Let \( \beta = \frac{r}{t} \)  
\[
\frac{\phi_1}{\phi_2} = 1 + \frac{1}{4\beta^2}
\]

(b) 

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \phi_1/\phi_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.0100</td>
</tr>
<tr>
<td>10</td>
<td>1.0025</td>
</tr>
<tr>
<td>20</td>
<td>1.0006</td>
</tr>
</tbody>
</table>

As the tube becomes thinner and \( \beta \) becomes larger, the ratio \( \phi_1/\phi_2 \) approaches unity. Thus, the thinnest the tube, the more accurate the approximate theory becomes.

**Problem 3.10-10** A thin-walled rectangular tube has uniform thickness \( t \) and dimensions \( a \times b \) to the median line of the cross section (see figure).

How does the shear stress in the tube vary with the ratio \( \beta = a/b \) if the total length \( L_m \) of the median line of the cross section and the torque \( T \) remain constant?

From your results, show that the shear stress is smallest when the tube is square (\( \beta = 1 \)).
Solution 3.10-10  Rectangular tube

$t =$ thickness (constant)
$a, b =$ dimensions of the tube

$\beta = \frac{a}{b}$

$L_m = 2(a + b) =$ constant

$T =$ constant

Shear stress

$\tau = \frac{T}{2tA_m}$  
$A_m = ab = \beta b^2$

$L_m = 2b(1 + \beta) =$ constant

$b = \frac{L_m}{2(1 + \beta)}$  
$A_m = \beta \left[ \frac{L_m}{2(1 + \beta)} \right]^2$

$A_m = \frac{\beta L_m^2}{4(1 + \beta)^2}$

$\tau = \frac{T}{2tA_m} = \frac{T(4(1 + \beta)^2}{2t\beta L_m^2} = \frac{2T(1 + \beta)^2}{tl^2_\beta}$

$T, t,$ and $L_m$ are constants.

Let $k = \frac{2T}{tl^2_m} =$ constant  
$\tau = k \frac{(1 + \beta)^2}{\beta}$

\[
\left(\frac{\tau}{k}\right)_{min} = 4 \quad \tau_{min} = \frac{8T}{tl^2_m}
\]

From the graph, we see that $\tau$ is minimum when $\beta = 1$

Alternate solution

$\tau = \frac{2T}{tL_m^2} \left( \frac{(1 + \beta)^2}{\beta} \right)$

\[
\frac{d\tau}{d\beta} = \frac{2T}{tL_m^2} \left[ \beta(2(1 + \beta) - (1 + \beta)^2(1) \right] = 0
\]

or $2\beta(1 + \beta) - (1 + \beta)^2 = 0 \quad : \beta = 1$

Thus, the tube is square and $\tau$ is either a minimum or a maximum. From the graph, we see that $\tau$ is a minimum.

Problem 3.10-11  A tubular aluminum bar $(G = 4 \times 10^6 \text{ psi})$ of square cross section (see figure) with outer dimensions 2 in. $\times$ 2 in. must resist a torque $T = 3000 \text{ lb-in}.$

Calculate the minimum required wall thickness $t_{min}$ if the allowable shear stress is 4500 psi and the allowable rate of twist is 0.01 rad/ft.
Solution 3.10-11  Square aluminum tube

Outer dimensions:
2.0 in. × 2.0 in.

\( G = 4 \times 10^6 \) psi

\( T = 3000 \) lb-in.

\( \tau_{\text{allow}} = 4500 \) psi

\( \theta_{\text{allow}} = 0.01 \) rad/ft = \( \frac{0.01}{12} \) rad/in.

Let \( b \) = outer dimension

\( = 2.0 \) in.

Centerline dimension \( = b - t \)

\( A_m = (b - t)^2 \)

\( L_m = 4(b - t) \)

\( J = \frac{4tA_m^2}{L_m} = \frac{4t(b - t)^4}{4(b - t)} = t(b - t)^3 \)

**Thicknesst \( t \) based upon shear stress**

\[ \tau = \frac{T}{2tA_m} \]

\[ tA_m = \frac{T}{2\tau} \]

\[ t(b - t)^2 = \frac{T}{2\tau} \]

Units: \( t \) = in. \( b \) = in. \( T \) = lb-in. \( \tau \) = psi

\[ t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3 \]

\[ 3t(2 - t)^2 - 1 = 0 \]

Solve for \( t \): \( t = 0.0915 \) in.

**Thicknesst \( t \) based upon rate of twist**

\[ \theta = \frac{T}{GJ} = \frac{T}{Gt(b - t)^3} \]

\[ t(b - t)^3 = \frac{T}{G\theta} \]

Units: \( t \) = in. \( G \) = psi \( \theta \) = rad/in.

\[ t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in.}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})} \]

\[ = \frac{9}{10} \]

\[ 10t(2 - t)^3 - 9 = 0 \]

Solve for \( t \):

\( t = 0.140 \) in.

**Angle of twist governs**

\( t_{\text{min}} = 0.140 \) in.

Problem 3.10-12  A thin tubular shaft of circular cross section (see figure) with inside diameter 100 mm is subjected to a torque of 5000 N · m.

If the allowable shear stress is 42 MPa, determine the required wall thickness \( t \) by using (a) the approximate theory for a thin-walled tube, and (b) the exact torsion theory for a circular bar.
Solution 3.10-12  Thin tube

\[ T = 5,000 \text{ N} \cdot \text{m} \quad d_1 = \text{inner diameter} = 100 \text{ mm} \]
\[ \tau_{\text{allow}} = 42 \text{ MPa} \]
\[ t \text{ is in millimeters.} \]
\[ r = \text{Average radius} \]
\[ = 50 \text{ mm} + \frac{t}{2} \]
\[ r_1 = \text{Inner radius} \]
\[ = 50 \text{ mm} \]
\[ r_2 = \text{Outer radius} \]
\[ = 50 \text{ mm} + t \quad A_m = \pi r^2 \]

(a) **Approximate theory**

\[ \tau = \frac{T}{2tA_m} = \frac{T}{2t(\pi r^2)} = \frac{T}{2\pi r^2 t} \]
\[ 42 \text{ MPa} = \frac{5,000 \text{ N} \cdot \text{m}}{2\pi \left(50 + \frac{t}{2}\right)^2 t} \]

or

\[ t \left(50 + \frac{t}{2}\right)^2 = \frac{5,000 \text{ N} \cdot \text{m}}{2\pi(42 \text{ MPa})} = \frac{5 \times 10^6}{84\pi} \text{ mm}^3 \]

Solve for \( t \):

\[ t = 6.66 \text{ mm} \]

(b) **Exact theory**

\[ \tau = \frac{Tr_2}{I_p} \]
\[ I_p = \frac{\tau}{2} (r_2^4 - r_1^4) = \frac{\tau}{2} [(50 + t)^4 - (50)^4] \]
\[ 42 \text{ MPa} = \frac{(5,000 \text{ N} \cdot \text{m})(50 + t)}{\frac{\tau}{2}[(50 + t)^4 - (50)^4]} \]
\[ (50 + t)^4 - (50)^4 = \frac{(5000 \text{ N} \cdot \text{m})(2)}{(\pi)(42 \text{ MPa})} \]
\[ = \frac{5 \times 10^6}{21\pi} \text{ mm}^3 \]

Solve for \( t \):

\[ t = 7.02 \text{ mm} \]

The approximate result is 5% less than the exact result. Thus, the approximate theory is nonconservative and should only be used for thin-walled tubes.
Problem 3.10-13  A long, thin-walled tapered tube $AB$ of circular cross section (see figure) is subjected to a torque $T$. The tube has length $L$ and constant wall thickness $t$. The diameter to the median lines of the cross sections at the ends $A$ and $B$ are $d_A$ and $d_B$, respectively.

Derive the following formula for the angle of twist of the tube:

$$\phi = \frac{2TL}{\pi Gt} \left( \frac{d_A + d_B}{d_A^2 d_B^2} \right)$$

**Hint:** If the angle of taper is small, we may obtain approximate results by applying the formulas for a thin-walled prismatic tube to a differential element of the tapered tube and then integrating along the axis of the tube.

Solution 3.10-13 Thin-walled tapered tube

For entire tube:

$$\phi = \frac{4T}{\pi GT} \int_0^L \frac{dx}{d_A + \left( \frac{d_B - d_A}{L} \right) x}^3$$

From table of integrals (see Appendix C):

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$\phi = \frac{4T}{\pi GT} \left[ -\frac{L}{2(d_B - d_A)d_B^2} + \frac{L}{2(d_B - d_A)d_A^2} \right]_0^L$$

$$\phi = \frac{2TL}{\pi Gt} \left( \frac{d_A + d_B}{d_A^2 d_B^2} \right)$$

For element of length $dx$:

$$d\phi = \frac{Tdx}{GJ(x)} = \frac{4Tdx}{G\pi t \left[ d_A + \left( \frac{d_B - d_A}{L} \right) x \right] ^3}$$
Stress Concentrations in Torsion

The problems for Section 3.11 are to be solved by considering the stress-concentration factors.

Problem 3.11-1  A stepped shaft consisting of solid circular segments having diameters $D_1 = 2.0 \text{ in.}$ and $D_2 = 2.4 \text{ in.}$ (see figure) is subjected to torques $T$. The radius of the fillet is $R = 0.1 \text{ in.}$

If the allowable shear stress at the stress concentration is 6000 psi, what is the maximum permissible torque $T_{\text{max}}$?

Solution 3.11-1  Stepped shaft in torsion

Use Fig. 3-48 for the stress-concentration factor

$K = 1.52 \quad \tau_{\text{max}} = K\tau_{\text{nom}} = K\left(\frac{16T_{\text{max}}}{\pi D_1^3}\right)$

$T_{\text{max}} = \frac{\pi D_1^3 \tau_{\text{max}}}{16K} = \frac{\pi(2.0 \text{ in.})^3(6000 \text{ psi})}{16(1.52)} = 6200 \text{ lb-in.}$

$\therefore T_{\text{max}} = 6200 \text{ lb-in.}$

Problem 3.11-2  A stepped shaft with diameters $D_1 = 40 \text{ mm}$ and $D_2 = 60 \text{ mm}$ is loaded by torques $T = 1100 \text{ N \cdot m}$ (see figure).

If the allowable shear stress at the stress concentration is 120 MPa, what is the smallest radius $R_{\text{min}}$ that may be used for the fillet?

Solution 3.11-2  Stepped shaft in torsion

Use Fig. 3-48 for the stress-concentration factor

$\tau_{\text{max}} = K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right)$

$K = \frac{\pi D_1^3 \tau_{\text{max}}}{16T} = \frac{\pi(40 \text{ mm})^3(120 \text{ MPa})}{16(1100 \text{ N \cdot m})} = 1.37$

$\frac{D_2}{D_1} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$

From Fig. 3-48 with $\frac{D_2}{D_1} = 1.5$ and $K = 1.37$, we get $R \approx 0.10$

$\therefore R_{\text{min}} = 0.10(40 \text{ mm}) = 4.0 \text{ mm}$
Problem 3.11-3 A full quarter-circular fillet is used at the shoulder of a stepped shaft having diameter $D_2 = 1.0$ in. (see figure). A torque $T = 500$ lb-in. acts on the shaft.

Determine the shear stress $\tau_{\text{max}}$ at the stress concentration for values as follows: $D_1$ is 0.7, 0.8, and 0.9 in. Plot a graph showing $\tau_{\text{max}}$ versus $D_1$.

Solution 3.11-3 Stepped shaft in torsion

\[ D_2 = 1.0 \text{ in.} \]
\[ T = 500 \text{ lb-in.} \]
\[ D_1 = 0.7, 0.8, \text{ and } 0.9 \text{ in.} \]

Full quarter-circular fillet ($D_2 = D_1 + 2R$)

\[ R = \frac{D_2 - D_1}{2} = 0.5 \text{ in.} - \frac{D_1}{2} \]

Use Fig. 3-48 for the stress-concentration factor

\[ \tau_{\text{max}} = K\tau_{\text{nom}} = K\left(\frac{16 T}{\pi D_1^3}\right) \]

\[ = K\frac{16(500 \text{ lb-in.})}{\pi D_1^3} = 2546 \frac{K}{D_1^3} \]

NOTE that $\tau_{\text{max}}$ gets smaller as $D_1$ gets larger, even though $K$ is increasing.

<table>
<thead>
<tr>
<th>$D_1$ (in.)</th>
<th>$D_2/D_1$</th>
<th>$R$ (in.)</th>
<th>$R/D_1$</th>
<th>$K$</th>
<th>$\tau_{\text{max}}$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.43</td>
<td>0.15</td>
<td>0.214</td>
<td>1.20</td>
<td>8900</td>
</tr>
<tr>
<td>0.8</td>
<td>1.25</td>
<td>0.10</td>
<td>0.125</td>
<td>1.29</td>
<td>6400</td>
</tr>
<tr>
<td>0.9</td>
<td>1.11</td>
<td>0.05</td>
<td>0.056</td>
<td>1.41</td>
<td>4900</td>
</tr>
</tbody>
</table>

Problem 3.11-4 The stepped shaft shown in the figure is required to transmit 600 kW of power at 400 rpm. The shaft has a full quarter-circular fillet, and the smaller diameter $D_1 = 100$ mm.

If the allowable shear stress at the stress concentration is 100 MPa, at what diameter $D_2$ will this stress be reached? Is this diameter an upper or a lower limit on the value of $D_2$?
Solution 3.11-4  Stepped shaft in torsion

Use the dashed line for a full quarter-circular fillet.

\[
\frac{R}{D_1} \approx 0.075 \quad R \approx 0.075 D_1 = 0.075 \times 100 \text{ mm} = 7.5 \text{ mm}
\]

\[
D_2 = D_1 + 2R = 100 \text{ mm} + 2(7.5 \text{ mm}) = 115 \text{ mm}
\]

\[
\therefore D_2 \approx 115 \text{ mm}
\]

This value of \(D_2\) is a lower limit.

(If \(D_2\) is less than 115 mm, \(R/D_1\) is smaller, \(K\) is larger, and \(\tau_{\text{max}}\) is larger, which means that the allowable stress is exceeded.)

Problem 3.11-5  A stepped shaft (see figure) has diameter \(D_2 = 1.5\) in. and a full quarter-circular fillet. The allowable shear stress is 15,000 psi and the load \(T = 4800\) lb-in.

What is the smallest permissible diameter \(D_1\)?
Solution 3.11-5  Stepped shaft in torsion

\[ D_2 = 1.5 \text{ in.} \]
\[ \tau_{\text{allow}} = 15,000 \text{ psi} \]
\[ T = 4800 \text{ lb-in.} \]

Full quarter-circular fillet \( D_2 = D_1 + 2R \)
\[
R = \frac{D_2 - D_1}{2} = 0.75 \text{ in.} - \frac{D_1}{2}
\]

Use Fig. 3-48 for the stress-concentration factor

\[
\tau_{\text{max}} = K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right)
\]

\[
= \frac{K}{D_1^3}\left[\frac{16(4800 \text{ lb-in.})}{\pi}\right]
\]

\[
= 24,450 \frac{K}{D_1^3}
\]

Use trial-and-error. Select trial values of \( D_1 \)

<table>
<thead>
<tr>
<th>( D_1 ) (in.)</th>
<th>( R ) (in.)</th>
<th>( R/D_1 )</th>
<th>( K )</th>
<th>( \tau_{\text{max}} ) (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.30</td>
<td>0.100</td>
<td>0.077</td>
<td>1.38</td>
<td>15,400</td>
</tr>
<tr>
<td>1.35</td>
<td>0.075</td>
<td>0.056</td>
<td>1.41</td>
<td>14,000</td>
</tr>
<tr>
<td>1.40</td>
<td>0.050</td>
<td>0.036</td>
<td>1.46</td>
<td>13,000</td>
</tr>
</tbody>
</table>

From the graph, minimum \( D_1 = 1.31 \) in.