9

Deflections of Beams

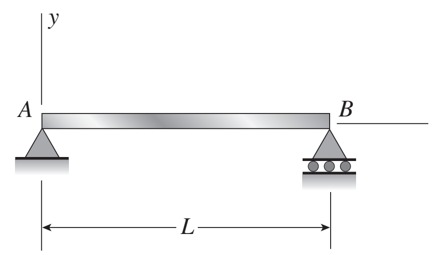
Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.2 have constant flexural rigidity EI.

Problem 9.2-1 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

Describe the load acting on the beam.



Probs. 9.2-1 and 9.2-2

Solution 9.2-1 Simple beam

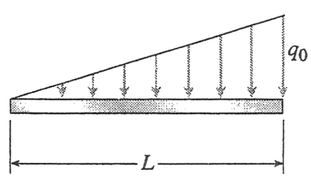
$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0 x}{LEI}$$

From Eq. (9-12c): $q = -EIv'''' = \dfrac{q_0 x}{L}$ $\leftarrow$

The load is a downward triangular load of maximum intensity q_0. $\leftarrow$



Problem 9.2-2 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and R_B at the supports.
(c) Determine the maximum bending moment M_{max}.

Solution 9.2-2 Simple beam

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI} \cos \frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI} \sin \frac{\pi x}{L}$$

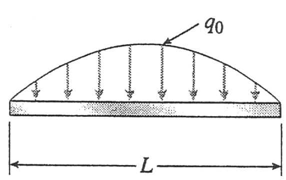
$$v''' = \frac{q_0 L}{\pi EI} \cos \frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI} \sin \frac{\pi x}{L}$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0 \sin \frac{\pi x}{L} \quad \leftarrow$$

The load has the shape of a sine curve, acts downward, and has maximum intensity q_0. ←



(b) REACTIONS (EQ. 9-12b)

$$V = EIv''' = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L}$$

$$\text{At } x = 0\text{: } V = R_A = \frac{q_0 L}{\pi} \quad \leftarrow$$

$$\text{At } x = L\text{: } V = -R_B = -\frac{q_0 L}{\pi};$$

$$R_B = \frac{q_0 L}{\pi} \quad \leftarrow$$

(c) MAXIMUM BENDING MOMENT (EQ. 9-12a)

$$M = EIv'' = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L}$$

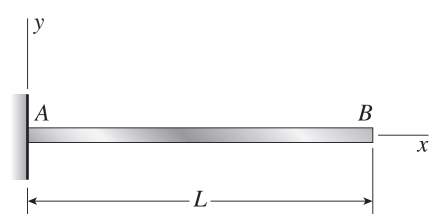
For maximum moment, $x = \frac{L}{2}$;

$$M_{max} = \frac{q_0 L^2}{\pi^2} \quad \leftarrow$$

Problem 9.2-3 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

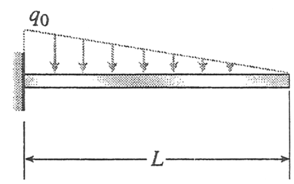
$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.



Probs. 9.2-3 and 9.2.-4

Solution 9.2-3 Cantilever beam



$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{LEI}(L - x)$$

From Eq. (9-12c):

$$q = -EIv'''' = q_0\left(1 - \frac{x}{L}\right) \quad \leftarrow$$

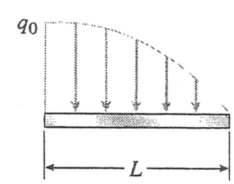
The load is a downward triangular load of maximum intensity q_0. $\leftarrow$

Problem 9.2-4 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

(a) Describe the load acting on the beam.
(b) Determine the reactions R_A and M_A at the support.

Solution 9.2-4 Cantilever beam



$$v = -\frac{q_0 x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

$$v' = -\frac{q_0}{60L^2EI}(15L^4x - 20L^3x^2 + 10L^2x^2 - x^5)$$

$$v'' = -\frac{q_0}{12L^2EI}(3L^4 - 8L^3x + 6L^2x^2 - x^4)$$

$$v''' = -\frac{q_0}{3L^2EI}(-2L^3 + 3L^2x - x^3)$$

$$v'''' = -\frac{q_0}{L^2EI}(L^2 - x^2)$$

(a) LOAD (EQ. 9-12c)

$$q = -EIv'''' = q_0\left(1 - \frac{x^2}{L^2}\right) \quad \leftarrow$$

The load is a downward parabolic load of maximum intensity q_0. $\leftarrow$

(b) REACTIONS R_A AND M_A (EQ. 9-12b AND EQ. 9-12a)

$$V = EIv''' = -\frac{q_0}{3L^2}(-2L^3 + 3L^2x - x^3)$$

$$\text{At } x = 0\text{: } V = R_A = \frac{2q_0L}{3} \quad \leftarrow$$

$$M = EIv'' = -\frac{q_0}{12L^2}(3L^4 - 8L^3x + 6L^2x^2 - x^4)$$

$$\text{At } x = 0\text{: } M = M_A = -\frac{q_0L^2}{4} \quad \leftarrow$$

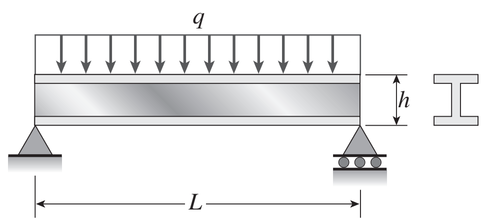
NOTE: Reaction R_A is positive upward.
Reaction M_A is positive clockwise (minus means M_A is counterclockwise).

Deflection Formulas

Problems 9.3-1 through 9.3-7 require the calculation of deflections using the formulas derived in Examples 9-1, 9-2, and 9-3. All beams have constant flexural rigidity EI.

Problem 9.3-1 A wide-flange beam (W 12 × 35) supports a uniform load on a simple span of length $L = 14$ ft (see figure).

Calculate the maximum deflection δ_{max} at the midpoint and the angles of rotation θ at the supports if $q = 1.8$ k/ft and $E = 30 \times 10^6$ psi. Use the formulas of Example 9-1.



Probs. 9.3-1 through 9.3-3

Solution 9.3-1 Simple beam (uniform load)

W 12 × 35 $\quad L = 14$ ft $= 168$ in.

$q = 1.8$ k/ft $= 150$ lb/in. $\quad E = 30 \times 10^6$ psi

$I = 285$ in.^{4}

MAXIMUM DEFLECTION (EQ. 9-18)

$$\delta_{max} = \frac{5\,qL^4}{384\,EI} = \frac{5(150\text{ lb/in.})(168\text{ in.})^4}{384(30 \times 10^6\text{ psi})(285\text{ in.}^4)}$$

$$= 0.182\text{ in.} \quad \leftarrow$$

ANGLE OF ROTATION AT THE SUPPORTS (EQS. 9-19 AND 9-20)

$$\theta = \theta_A = \theta_B = \frac{qL^3}{24\,EI}$$

$$= \frac{(150\text{ lb/in.})(168\text{ in.})^3}{24(30 \times 10^6\text{ psi})(285\text{ in.}^4)}$$

$$= 0.003466\text{ rad} = 0.199° \quad \leftarrow$$

Problem 9.3-2 A uniformly loaded steel wide-flange beam with simple supports (see figure) has a downward deflection of 10 mm at the midpoint and angles of rotation equal to 0.01 radians at the ends.

Calculate the height h of the beam if the maximum bending stress is 90 MPa and the modulus of elasticity is 200 GPa. (*Hint*: Use the formulas of Example 9-1.)

Solution 9.3-2 Simple beam (uniform load)

$\delta = \delta_{max} = 10$ mm $\quad \theta = \theta_A = \theta_B = 0.01$ rad

$\sigma = \sigma_{max} = 90$ MPa $\quad E = 200$ GPa

Calculate the height h of the beam.

$$\text{Eq. (9-18): } \delta = \delta_{max} = \frac{5qL^4}{384EI} \text{ or } q = \frac{384EI\delta}{5L^4} \quad (1)$$

$$\text{Eq. (9-19): } \theta = \theta_A = \frac{qL^3}{24EI} \text{ or } q = \frac{24EI\theta}{L^3} \quad (2)$$

$$\text{Equate (1) and (2) and solve for } L\text{: } L = \frac{16\delta}{5\theta} \quad (3)$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \qquad \therefore \sigma = \frac{qL^2h}{16I} \quad (4)$$

$$\text{Solve Eq. (4) for } h\text{: } h = \frac{16I\sigma}{qL^2} \quad (5)$$

Substitute for q from (2) and for L from (3):

$$h = \frac{32\sigma\delta}{15E\theta^2} \quad \leftarrow$$

Substitute numerical values:

$$h = \frac{32(90\text{ MPa})(10\text{ mm})}{15(200\text{ GPa})(0.01\text{ rad})^2} = 96\text{ mm} \quad \leftarrow$$

Problem 9.3-3 What is the span length L of a uniformly loaded simple beam of wide-flange cross section (see figure) if the maximum bending stress is 12,000 psi, the maximum deflection is 0.1 in., the height of the beam is 12 in., and the modulus of elasticity is 30×10^6 psi? (Use the formulas of Example 9-1.)

Solution 9.3-3 Simple beam (uniform load)

$\sigma = \sigma_{max} = 12{,}000$ psi $\qquad \delta = \delta_{max} = 0.1$ in.

$h = 12$ in. $\qquad E = 30 \times 10^6$ psi

Calculate the span length L.

Eq. (9-18): $\delta = \delta_{max} = \dfrac{5qL^4}{384EI}$ or $q = \dfrac{384EI\delta}{5L^4}$ (1)

Flexure formula: $\sigma = \dfrac{Mc}{I} = \dfrac{Mh}{2I}$

Maximum bending moment:

$M = \dfrac{qL^2}{8} \qquad \therefore \sigma = \dfrac{qL^2h}{16I}$ (2)

Solve Eq. (2) for q: $q = \dfrac{16I\sigma}{L^2h}$ (3)

Equate (1) and (2) and solve for L:

$L^2 = \dfrac{24\,Eh\delta}{5\sigma} \qquad L = \sqrt{\dfrac{24\,Eh\delta}{5\sigma}}$ ←

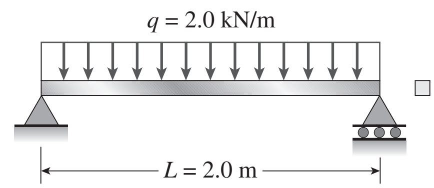
Substitute numerical values:

$$L^2 = \frac{24(30 \times 10^6 \text{ psi})(12 \text{ in.})(0.1 \text{ in.})}{5(12{,}000 \text{ psi})} = 14{,}400 \text{ in.}^2$$

$L = 120$ in. $= 10$ ft ←

Problem 9.3-4 Calculate the maximum deflection δ_{max} of a uniformly loaded simple beam (see figure) if the span length $L = 2.0$ m, the intensity of the uniform load $q = 2.0$ kN/m, and the maximum bending stress $\sigma = 60$ MPa.

The cross section of the beam is square, and the material is aluminum having modulus of elasticity $E = 70$ GPa. (Use the formulas of Example 9-1.)



Solution 9.3-4 Simple beam (uniform load)

$L = 2.0$ m $\qquad q = 2.0$ kN/m

$\sigma = \sigma_{max} = 60$ MPa $\qquad E = 70$ GPa

Cross section (square; b = width)

$I = \dfrac{b^4}{12} \qquad S = \dfrac{b^3}{6}$

Maximum deflection (Eq. 9-18): $\delta = \dfrac{5qL^4}{384\,EI}$ (1)

Substitute for I: $\delta = \dfrac{5qL^4}{32\,Eb^4}$ (2)

Flexure formula with $M = \dfrac{qL^2}{8}$: $\sigma = \dfrac{M}{S} = \dfrac{qL^2}{8S}$

Substitute for S: $\sigma = \dfrac{3qL^2}{4b^3}$ (3)

Solve for b^3: $b^3 = \dfrac{3qL^2}{4\sigma}$ (4)

Substitute b into Eq. (2): $\delta_{max} = \dfrac{5L\sigma}{24E}\left(\dfrac{4L\sigma}{3q}\right)^{1/3}$ ←

(The term in parentheses is nondimensional.)

Substitute numerical values:

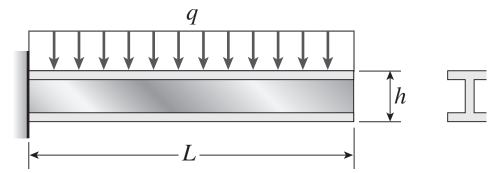
$$\frac{5L\sigma}{24E} = \frac{5(2.0 \text{ m})(60 \text{ MPa})}{24(70 \text{ GPa})} = \frac{1}{2800} \text{ m} = \frac{1}{2.8} \text{ mm}$$

$$\left(\frac{4\,L\sigma}{3q}\right)^{1/3} = \left[\frac{4(2.0 \text{ m})(60 \text{ MPa})}{3(2000 \text{ N/m})}\right]^{1/3} = 10(80)^{1/3}$$

$\delta_{max} = \dfrac{10(80)^{1/3}}{2.8}$ mm $= 15.4$ mm ←

Problem 9.3-5 A cantilever beam with a uniform load (see figure) has a height h equal to 1/8 of the length L. The beam is a steel wideflange section with $E = 28 \times 10^6$ psi and an allowable bending stress of 17,500 psi in both tension and compression.

Calculate the ratio δ/L of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load. (Use the formulas of Example 9-2.)



Solution 9.3-5 Cantilever beam (uniform load)

$$\frac{h}{L} = \frac{1}{8} \qquad E = 28 \times 10^6 \text{ psi} \qquad \sigma = 17{,}500 \text{ psi}$$

Calculate the ratio δ/L.

Maximum deflection (Eq. 9-26): $\delta_{max} = \dfrac{qL^4}{8EI}$ (1)

$$\therefore \frac{\delta}{L} = \frac{qL^3}{8EI} \qquad (2)$$

Flexure formula with $M = \dfrac{qL^2}{2}$:

$$\sigma = \frac{Mc}{I} = \left(\frac{qL^2}{2}\right)\left(\frac{h}{2I}\right) = \frac{qL^2h}{4I}$$

Solve for q:

$$q = \frac{4I\sigma}{L^2h} \qquad (3)$$

Substitute q from (3) into (2):

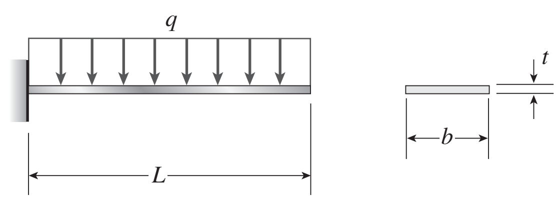
$$\frac{\delta}{L} = \frac{\sigma}{2E}\left(\frac{L}{h}\right) \quad \leftarrow$$

Substitute numerical values:

$$\frac{\delta}{L} = \frac{17{,}500 \text{ psi}}{2(28 \times 10^6 \text{ psi})}(8) = \frac{1}{400} \quad \leftarrow$$

Problem 9.3-6 A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 27.5$ μm and rectangular cross section of width $b = 4.0$ μm and thickness $t = 0.88$ μm. The total load on the beam is 17.2 μN.

If the deflection at the end of the beam is 2.46 μm, what is the modulus of elasticity E_g of the gold alloy? (Use the formulas of Example 9-2.)



Solution 9.3-6 Gold-alloy microbeam

Cantilever beam with a uniform load.

$L = 27.5$ μm $\qquad b = 4.0$ μm $\qquad t = 0.88$ μm

$qL = 17.2$ μN $\qquad \delta_{max} = 2.46$ μm

Determine E_g.

Eq. (9-26): $\delta = \dfrac{qL^4}{8E_gI}$ or $E_g = \dfrac{qL^4}{8I\delta_{max}}$

$$I = \frac{bt^3}{12} \qquad E_g = \frac{3qL^4}{2bt^3\delta_{max}} \quad \leftarrow$$

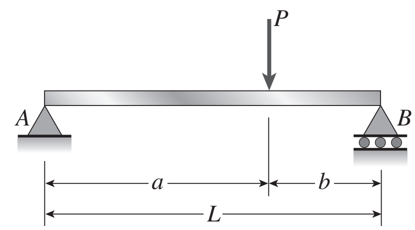
Substitute numerical values:

$$E_g = \frac{3(17.2 \text{ mN})(27.5 \text{ mm})^3}{2(4.0 \text{ mm})(0.88 \text{ mm})^3(2.46 \text{ mm})}$$

$$= 80.02 \times 10^9 \text{ N/m}^2 \quad \text{or} \quad E_g = 80.0 \text{ GPa} \quad \leftarrow$$

Problem 9.3-7 Obtain a formula for the ratio δ_C/δ_{max} of the deflection at the midpoint to the maximum deflection for a simple beam supporting a concentrated load P (see figure).

From the formula, plot a graph of δ_C/δ_{max} versus the ratio a/L that defines the position of the load ($0.5 < a/L < 1$). What conclusion do you draw from the graph? (Use the formulas of Example 9-3.)



Solution 9.3-7 Simple beam (concentrated load)

Eq. (9-35): $\delta_C = \dfrac{Pb(3L^2 - 4b^2)}{48EI} \quad (a \geq b)$

Eq. (9-34): $\delta_{max} = \dfrac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}LEI} \quad (a \geq b)$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(3L^2 - 4b^2)}{16(L^2 - b^2)^{3/2}} \quad (a \geq b)$$

Replace the distance b by the distance a by substituting $L - a$ for b:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)(-L^2 + 8aL - 4a^2)}{16(2aL - a^2)^{3/2}}$$

Divide numerator and denominator by L^2:

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16L\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}}$$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}} \quad \leftarrow$$

ALTERNATIVE FORM OF THE RATIO

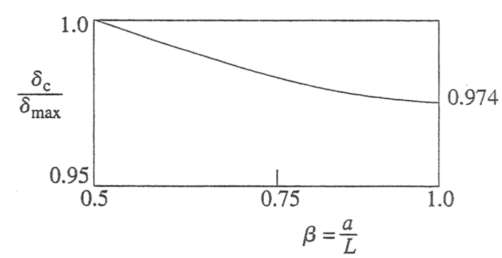
Let $\beta = \dfrac{a}{L}$

$$\frac{\delta_c}{\delta_{max}} = \frac{(3\sqrt{3})(-1 + 8\beta - 4\beta^2)}{16(2\beta - \beta^2)^{3/2}} \quad \leftarrow$$

GRAPH OF δ_c/δ_{max} VERSUS $\beta = a/L$

Because $a \geq b$, the ratio β versus from 0.5 to 1.0.

β	$\dfrac{\delta_c}{\delta_{max}}$
0.5	1.0
0.6	0.996
0.7	0.988
0.8	0.981
0.9	0.976
1.0	0.974

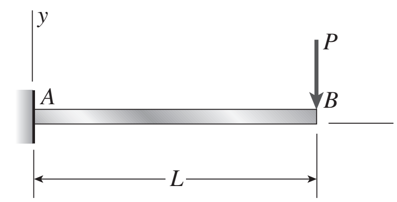


NOTE: The deflection δ_c at the midpoint of the beam is almost as large as the maximum deflection δ_{max}. The greatest difference is only 2.6% and occurs when the load reaches the end of the beam ($\beta = 1$).

Deflections by Integration of the Bending-Moment Equation

Problems 9.3-8 through 9.3-16 are to be solved by integrating the second-order differential equation of the deflection curve (the bending-moment equation). The origin of coordinates is at the left-hand end of each beam, and all beams have constant flexural rigidity EI.

Problem 9.3-8 Derive the equation of the deflection curve for a cantilever beam AB supporting a load P at the free end (see figure). Also, determine the deflection δ_B and angle of rotation θ_B at the free end. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-8 Cantilever beam (concentrated load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -P(L - x)$$

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

B.C. $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

$$v = -\frac{Px^2}{6EI}(3L - x) \quad \leftarrow$$

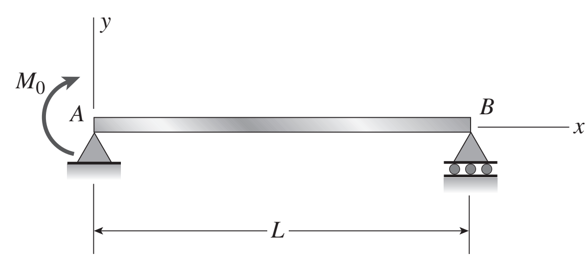
$$v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{PL^3}{3EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{PL^2}{2EI} \quad \leftarrow$$

(These results agree with Case 4, Table G-1.)

Problem 9.3-9 Derive the equation of the deflection curve for a simple beam AB loaded by a couple M_0 at the left-hand support (see figure). Also, determine the maximum deflection δ_{max}. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-9 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \quad \therefore C_2 = 0$

B.C. $v(L) = 0 \quad \therefore C_1 = -\frac{M_0L}{3}$

$$v = -\frac{M_0x}{6LEI}(2L^2 - 3Lx + x^2) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$v' = -\frac{M_0}{6LEI}(2L^2 - 6Lx + 3x^2)$$

Set $v' = 0$ and solve for x:

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \longleftarrow$$

Substitute x_1 into the equation for v:

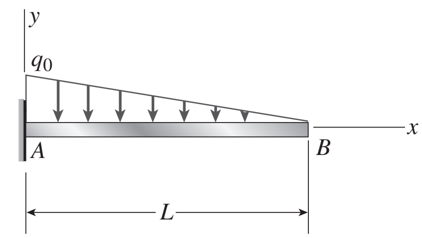
$$\delta_{\max} = -(v)_{x=x_1}$$

$$= \frac{M_0 L^2}{9\sqrt{3}EI} \quad \longleftarrow$$

(These results agree with Case 7, Table G-2.)

Problem 9.3-10 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-10 Cantilever beam (triangular load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q_0}{6L}(L - x)^3$$

$$EIv' = \frac{q_0}{24L}(L - x)^4 + C_1$$

B.C. $v'(0) = 0 \qquad \therefore C_1 = -\frac{q_0 L^3}{24}$

$$EIv = -\frac{q_0}{120L}(L - x)^5 - \frac{q_0 L^3 x}{24} + C_2$$

B.C. $v(0) = 0 \qquad \therefore C_2 = \frac{q_0 L^4}{120}$

$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3) \quad \longleftarrow$$

$$v' = -\frac{q_0 x}{24LEI}(4L^3 - 6L^2x + 4Lx^2 - x^3)$$

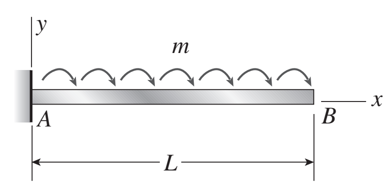
$$\delta_B = -v(L) = \frac{q_0 L^4}{30EI} \quad \longleftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24EI} \quad \longleftarrow$$

(These results agree with Case 8, Table G-1.)

Problem 9.3-11 A cantilever beam AB is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity m per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-11 Cantilever beam (distributed moment)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -m(L - x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

B.C. $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

B.C. $v(0) = 0 \qquad \therefore C_2 = 0$

$$v = -\frac{mx^2}{6\,EI}(3L - x) \qquad \leftarrow$$

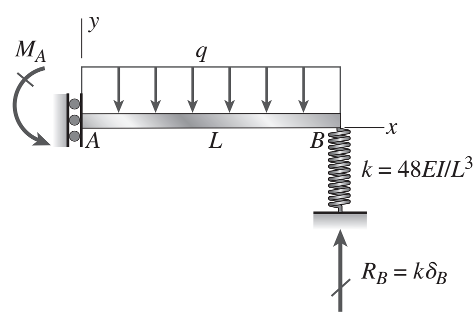
$$v' = -\frac{mx}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{mL^3}{3\,EI} \qquad \leftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2EI} \qquad \leftarrow$$

Problem 9.3-12 The beam shown in the figure has a guided support at A and a spring support at B. The guided support permits vertical movement but no rotation.

Derive the equation of the deflection curve and determine the deflection δ_B at end B due to the uniform load of intensity q. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-12

BENDING-MOMENT EQUATION

$$EIv'' = M(x) = \frac{qL^2}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qL^2x}{2} - \frac{qx^3}{24} + C_1$$

$$EIv' = \frac{qL^2x^2}{2} - \frac{qx^4}{24} + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

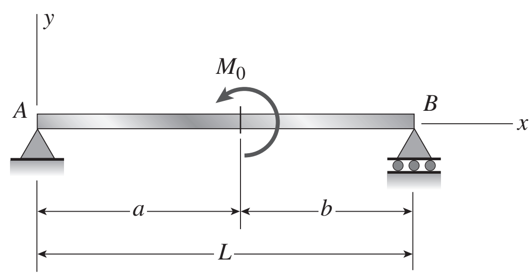
B.C. $v(L) = \dfrac{qL}{k} = -\dfrac{qL^4}{48EI} \qquad C_2 = -\dfrac{11qL^4}{48}$

$$v(x) = -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4) \qquad \leftarrow$$

$$\delta_B = -v(L) = \frac{qL^4}{48EI} \qquad \leftarrow$$

Note that $R_B = k\delta_B = qL$ which agrees with $\sum F_{\text{vert}} = 0$

Problem 9.3-13 Derive the equations of the deflection curve for a simple beam AB loaded by a couple M_0 acting at distance a from the left-hand support (see figure). Also, determine the deflection δ_0 at the point where the load is applied. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-13 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = \frac{M_0 x}{L} \quad (0 \le x \le a)$$

$$EIv' = \frac{M_0 x^2}{2L} + C_1 \quad (0 \le x \le a)$$

$$EIv'' = M = -\frac{M_0}{L}(L - x) \quad (a \le x \le L)$$

$$EIv' = -\frac{M_0}{L}\left(Lx - \frac{x^2}{2}\right) + C_2 \quad (a \le x \le L)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$$\therefore C_2 = C_1 + M_0 a$$

$$EIv = \frac{M_0 x^3}{6L} + C_1 x + C_3 \quad (0 \le x \le a)$$

B.C. 2 $v(0) = 0$ $\quad \therefore C_3 = 0$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_1 x + M_0 a x + C_4$$

$$(a \le x \le L)$$

B.C. 3 $v(L) = 0$ $\quad \therefore C_4 = -M_0 L\left(a - \frac{L}{3}\right) - C_1 L$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = -\frac{M_0 a^2}{2}$$

$$C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

$$v = -\frac{M_0 x}{6\,LEI}(6aL - 3a^2 - 2L^2 - x^2)$$

$$(0 \le x \le a) \quad \longleftarrow$$

$$v = -\frac{M_0}{6\,LEI}(3a^2L - 3a^2x - 2L^2x + 3Lx^2 - x^3)$$

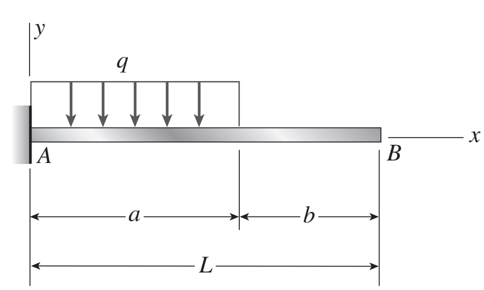
$$(a \le x \le L) \quad \longleftarrow$$

$$\delta_0 = -v(a) = \frac{M_0 a(L - a)(2a - L)}{3LEI}$$

$$= \frac{M_0 ab(2a - L)}{3LEI} \quad \longleftarrow$$

NOTE: δ_0 is positive downward. The preceding results agree with Case 9, Table G-2.

Problem 9.3-14 Derive the equations of the deflection curve for a cantilever beam AB carrying a uniform load of intensity q over part of the span (see figure). Also, determine the deflection δ_B at the end of the beam. (*Note:* Use the second-order differential equation of the deflection curve.)



Solution 9.3-14 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-12a)

$$EIv'' = M = -\frac{q}{2}(a - x)^2 = -\frac{q}{2}(a^2 - 2ax + x^2)$$

$$(0 \le x \le a)$$

$$EIv' = -\frac{q}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) + C_1 \quad (0 \le x \le a)$$

B.C. 1 $v'(0) = 0 \qquad \therefore C_1 = 0$

$$EIv'' = M = 0 \quad (a \le x \le L)$$

$$EIv' = C_2 \qquad (a \le x \le L)$$

B.C. 2 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a \qquad \therefore C_2 = -\frac{qa^3}{6}$

$$EIv = -\frac{q}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_3 \quad (0 \le x \le a)$$

B.C. 3 $v(0) = 0 \qquad \therefore C_3 = 0$

$$EIv = C_2x + C_4 = -\frac{qa^3x}{6} + C_4 \quad (a \le x \le L)$$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a \qquad \therefore C_4 = \frac{qa^4}{24}$

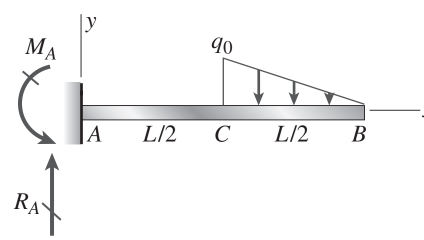
$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \quad (0 \le x \le a) \quad \leftarrow$$

$$v = -\frac{qa^3}{24EI}(4x - a) \quad (a \le x \le L) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{qa^3}{24EI}(4L - a) \quad \leftarrow$$

(These results agree with Case 2, Table G-1.)

Problem 9.3-15 Derive the equations of the deflection curve for a cantilever beam AB supporting a distributed load of peak intensity q_0 acting over one-half of the length (see figure). Also, obtain formulas for the deflections δ_B and δ_C at points B and C, respectively. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-15

BENDING-MOMENT EQUATION

For $0 \le x \le \frac{L}{2}$

$$EIv'' = M(x) = \frac{q_0Lx}{4} - \frac{q_0L^2}{6}$$

$$EIv' = \frac{q_0Lx^2}{8} - \frac{q_0L^2x}{6} + C_1$$

$$EIv = \frac{q_0Lx^3}{24} - \frac{q_0L^2x^2}{12} + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(0) = 0 \qquad C_2 = 0$

$$v'\left(\frac{L}{2}\right) = -\frac{5q_0L^3}{96EI}$$

$$v(x) = \frac{q_0L}{24EI}(x^3 - 2Lx^2) \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{q_0L^4}{64EI} \quad \leftarrow$$

For $\frac{L}{2} \le x \le L$

$$EIv'' = M(x) = \frac{q_0Lx}{4} - \frac{q_0L^2}{6} - \frac{q_0}{L}(L - x)\left(x - \frac{L}{2}\right)^2 - \frac{1}{2}\left[q_0 - \frac{2q_0}{L}(L - x)\right]\left(x - \frac{L}{2}\right)^2\frac{2}{3}$$

$$EIv'' = M(x) = \frac{-q_0}{3L}(-3L^2x + L^3 + 3Lx^2 - x^3)$$

$$EIv' = -\frac{q_0}{3L}\left(\frac{-3}{2}L^2x^2 + L^3x + Lx^3 - \frac{x^4}{4}\right) + C_3$$

$$EIv = -\frac{q_0}{3L}\left(\frac{-1}{2}L^2x^3 + \frac{1}{2}L^3x^2 + \frac{1}{4}Lx^4 - \frac{1}{20}x^5\right) + C_3x + C_4$$

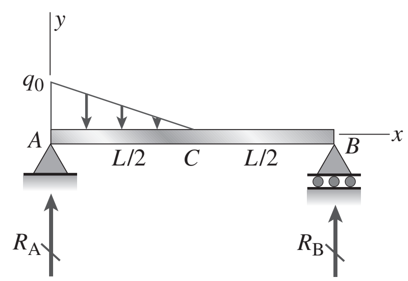
B.C. $v'\left(\frac{L}{2}\right) = -\frac{5q_0L^3}{96EI}$ $\qquad C_3 = \frac{5}{192}q_0L^3$

B.C. $v\left(\frac{L}{2}\right) = -\frac{q_0L^4}{64EI}$ $\qquad C_4 = \frac{-1}{320}q_0L^4$

$$v(x) = \frac{-q_0}{960LEI}(-160L^2x^3 + 160L^3x^2 + 80Lx^4 - 16x^5 - 25L^4x + 3L^5) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{7q_0L^4}{160EI} \quad \leftarrow$$

Problem 9.3-16 Derive the equations of the deflection curve for a simple beam AB with a distributed load of peak intensity q_0 acting over the left-hand half of the span (see figure). Also, determine the deflection δ_C at the midpoint of the beam. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-16

BENDING-MOMENT EQUATION

For $0 \le x \le \frac{L}{2}$

$$EIv'' = M(x) = \frac{5q_0Lx}{24} - \frac{2q_0}{L}\left(\frac{L}{2} - x\right)\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[q_0 - \frac{2q_0}{L}\left(\frac{L}{2} - x\right)\right]x\frac{2}{3}x$$

$$EIv'' = \frac{q_0}{24L}(5L^2x - 12x^2L + 8x^3)$$

$$EIv' = \frac{q_0}{24L}\left(\frac{5L^2x^2}{2} - 4x^3L + 2x^4\right) + C_1 \qquad (1)$$

$$EIv = \frac{q_0}{24L}\left(\frac{5L^2x^3}{6} - x^4L + \frac{2x^5}{5}\right) + C_1x + C_2$$

B.C. $v(0) = 0 \qquad C_2 = 0$

$$EIv = \frac{q_0}{24L}\left(\frac{5L^2x^3}{6} - x^4L + \frac{2x^5}{5}\right) + C_1x \qquad (2)$$

For $\frac{L}{2} \le x \le L$

$$EIv'' = M(x) = \frac{5q_0Lx}{24} - \frac{1}{2}q_0\frac{L}{2}\left(x - \frac{L}{6}\right)$$

$$EIv'' = \frac{Lq_0}{24}(-x + L)$$

$$EIv' = \frac{Lq_0}{24}\left(\frac{-x^2}{2} + Lx\right) + C_3 \qquad (3)$$

$$EIv = \frac{Lq_0}{24}\left(\frac{-x^3}{6} + \frac{Lx^2}{2}\right) + C_3x + C_4 \quad (4)$$

B.C. $v(L) = 0 \qquad 0 = \frac{q_0L^4}{72} + C_3L + C_4 \quad (5)$

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right)$

$$\frac{1}{96}q_0L^3 + C_1 = \frac{1}{64}q_0L^3 + C_3 \quad (6)$$

B.C. $v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$

$$\frac{13}{5760}q_0L^4 + \frac{1}{2}C_1L = \frac{5}{1152}q_oL^4 + \frac{1}{2}C_3L + C_4 \quad (7)$$

From (5)–(7)

$$C_1 = \frac{-53}{5760}q_0L^3 \qquad C_3 = \frac{-83}{5760}q_0L^3$$

$$C_4 = \frac{1}{1920}q_0L^4$$

For $0 \le x \le \frac{L}{2}$

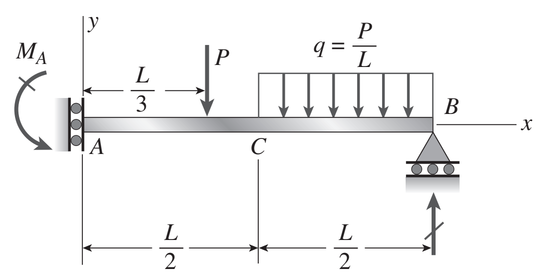
$$v(x) = \frac{q_0x}{5760LEI}(200x^2L^2 - 240x^3L + 96x^4 - 53L^4) \quad \longleftarrow$$

For $\frac{L}{2} \le x \le L$

$$v(x) = \frac{-Lq_0}{5760EI}(40x^3 - 120Lx^2 + 83L^2x - 3L^3) \quad \longleftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{3q_0L^4}{1280EI} \quad \longleftarrow$$

Problem 9.3-17 The beam shown in the figure has a guided support at A and a roller support at B. The guided support permits vertical movement but no rotation. Derive the equation of the deflection curve and determine the deflection δ_A at end A and also δ_C at point C due to the uniform load of intensity $q = P/L$ applied over segment CB and load P at $x = L/3$. (*Note*: Use the second-order differential equation of the deflection curve.)



Solution 9.3-17

BENDING-MOMENT EQUATION

For $0 \le x \le \frac{L}{3} \qquad EIv'' = M(x) = \frac{19}{24}PL$

$$EIv' = \frac{19}{24}PLx + C_1$$

$$EIv = \frac{19}{48}PLx^2 + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0 \qquad EIv' = \frac{19}{24}PLx \qquad EIv = \frac{19}{48}PLx^2 + C_2$

For $\frac{L}{3} \le x \le \frac{L}{2} \qquad EIv'' = M(x) = \frac{19}{24}PL - P\left(x - \frac{L}{3}\right)$

$$EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3}$$

$$EIv' = \frac{19}{24}PLx - \frac{Px^2}{2} + \frac{PLx}{3} + C_3$$

$$EIv = \frac{19}{48}PLx^2 - \frac{Px^3}{6} + \frac{PLx^2}{6} + C_3x + C_4$$

For $\frac{L}{2} \le x \le L$ $\quad EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3} - \frac{P}{L}\left(x - \frac{L}{2}\right)^2\frac{1}{2}$

$$EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3} - \frac{Px^2}{2L} + \frac{Px}{2} - \frac{PL}{8}$$

$$EIv' = \frac{19}{24}PLx - \frac{Px^2}{2} + \frac{PLx}{3} - \frac{Px^3}{6L} + \frac{Px^2}{4} - \frac{PLx}{8} + C_5$$

$$EIv = \frac{19}{48}PLx^2 - \frac{Px^3}{6} + \frac{PLx^2}{6} - \frac{Px^4}{24L} + \frac{Px^3}{12} - \frac{PLx^2}{16} + C_5x + C_6$$

B.C. $v(L) = 0 \quad 0 = \frac{19}{48}PLL^2 - \frac{PL^3}{6} + \frac{PLL^2}{6} - \frac{PL^4}{24L} + \frac{PL^3}{12} - \frac{PLL^2}{16} + C_5L + C_6$ (1)

B.C. $v'_L\left(\frac{L}{3}\right) = v'_R\left(\frac{L}{3}\right) \quad 0 = -\frac{P\left(\frac{L}{3}\right)^2}{2} + \frac{PL\left(\frac{L}{3}\right)}{3} + C_3$ (2)

B.C. $v_L\left(\frac{L}{3}\right) = v_R\left(\frac{L}{3}\right) \quad C_2 = -\frac{P\left(\frac{L}{3}\right)^3}{6} + \frac{PL\left(\frac{L}{3}\right)^2}{6} + C_3\left(\frac{L}{3}\right) + C_4$ (3)

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right) \quad C_3 = -\frac{P\left(\frac{L}{2}\right)^3}{6L} + \frac{P\left(\frac{L}{2}\right)^2}{4} - \frac{PL\left(\frac{L}{2}\right)}{8} + C_5$ (4)

B.C. $v_L(a) = v_R(a) \quad C_3\frac{L}{2} + C_4 = -\frac{P\left(\frac{L}{2}\right)^4}{24L} + \frac{P\left(\frac{L}{2}\right)^3}{12} - \frac{PL\left(\frac{L}{2}\right)^2}{16} + C_5\left(\frac{L}{2}\right) + C_6$ (5)

From (1)–(5)

$C_2 = \frac{-3565}{10368}PL^3 \quad C_3 = \frac{-1}{18}PL^2 \quad C_4 = \frac{-389}{1152}PL^3 \quad C_5 = \frac{-5}{144}PL^2 \quad C_6 = \frac{-49}{144}PL^3$

For $0 \le x \le \frac{L}{3}$ $\quad v(x) = \frac{-PL}{10368EI}(-4104x^2 + 3565L^2) \quad \leftarrow$

For $\frac{L}{3} \le x \le \frac{L}{2}$ $\quad v(x) = \frac{-P}{1152EI}(-648Lx^2 + 192x^3 + 64L^2x + 389L^3) \quad \leftarrow$

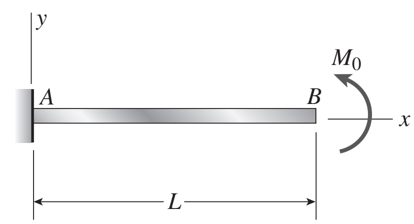
For $\frac{L}{2} \le x \le L$ $\quad v(x) = \frac{-P}{144EIL}(-72L^2x^2 + 12x^3L + 6x^4 + 5L^3x + 49L^4) \quad \leftarrow$

$\delta_A = -v(0) = \frac{3565PL^3}{10368EI} \quad \leftarrow \qquad \delta_C = -v\left(\frac{L}{3}\right) = \frac{3109PL^3}{10368EI} \quad \leftarrow$

Deflections by Integration of the Shear Force and Load Equations

The beams described in the problems for Section 9.4 have constant flexural rigidity EI. Also, the origin of coordinates is at the left-hand end of each beam.

Problem 9.4-1 Derive the equation of the deflection curve for a cantilever beam AB when a couple M_0 acts counterclockwise at the free end (see figure). Also, determine the deflection δ_B and slope θ_B at the free end. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-1 Cantilever beam (couple M_0)

SHEAR-FORCE EQUATION (EQ. 9-12b).

$$EIv''' = V = 0$$

$$EIv'' = C_1$$

B.C. 1 $M = M_0$ $\quad EIv'' = M = M_0 = C_1$

$$EIv' = C_1x + C_2 = M_0x + C_2$$

B.C. 2 $v'(0) = 0 \qquad \therefore C_2 = 0$

$$EIv = \frac{M_0x^2}{2} + C_3$$

B.C. 3 $v(0) = 0 \qquad \therefore C_3 = 0$

$$v = \frac{M_0x^2}{2EI} \quad \leftarrow$$

$$v' = \frac{M_0x}{EI}$$

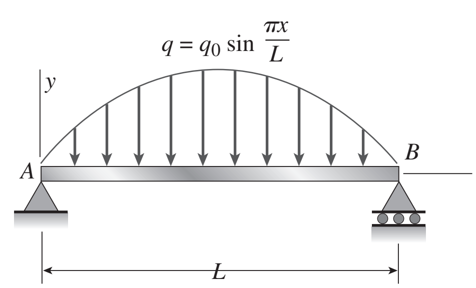
$$\delta_B = v(L) = \frac{M_0L^2}{2EI} \text{ (upward)} \quad \leftarrow$$

$$\theta_B = v'(L) = \frac{M_0L}{EI} \text{ (counterclockwise)} \quad \leftarrow$$

(These results agree with Case 6, Table G-1.)

Problem 9.4-2 A simple beam AB is subjected to a distributed load of intensity $q = q_0 \sin \pi x/L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_{max} at the midpoint of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-2 Simple beam (sine load)

LOAD EQUATION (EQ. 9-12c).

$$EIv'''' = -q = -q_0 \sin\frac{\pi x}{L}$$

$$EIv''' = q_0\left(\frac{L}{\pi}\right)\cos\frac{\pi x}{L} + C_1$$

$$EIv'' = q_0\left(\frac{L}{\pi}\right)^2 \sin\frac{\pi x}{L} + C_1x + C_2$$

B.C. 1 $EIv'' = M \qquad EIv''(0) = 0 \qquad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \qquad \therefore C_1 = 0$

$$EIv' = -q_0\left(\frac{L}{\pi}\right)^3 \cos\frac{\pi x}{L} + C_3$$

$$EIv = -q_0\left(\frac{L}{\pi}\right)^4 \sin\frac{\pi x}{L} + C_3x + C_4$$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_3 = 0$

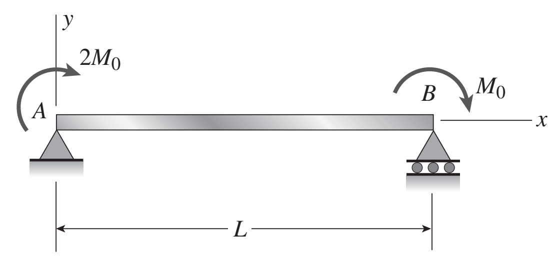
$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L} \quad \leftarrow$$

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{q_0 L^4}{\pi^4 EI} \quad \leftarrow$$

(These results agree with Case 13, Table G-2.)

Problem 9.4-3 The simple beam AB shown in the figure has moments $2M_0$ and M_0 acting at the ends.

Derive the equation of the deflection curve, and then determine the maximum deflection $\delta_{\max}$. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-3 Simple beam with two couples

Reaction at support A: $R_A = \frac{3M_0}{L}$ (downward)

Shear force in beam: $V = -R_A = -\frac{3M_0}{L}$

SHEAR-FORCE EQUATION (EQ. 9-12b)

$$EIv''' = V = -\frac{3M_0}{L}$$

$$EIv'' = -\frac{3M_0 x}{L} + C_1$$

B.C. 1 $EIv'' = M \qquad EIv''(0) = 2M_0 \qquad \therefore C_1 = 2M_0$

$$EIv' = -\frac{3M_0 x^2}{2L} + 2M_0 x + C_2$$

$$EIv = -\frac{M_0 x^3}{2L} + M_0 x^2 + C_2 x + C_3$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(L) = 0 \quad \therefore C_2 = -\frac{M_0 L}{2}$

$$v = -\frac{M_0 x}{2LEI}(L^2 - 2Lx + x^2)$$

$$= -\frac{M_0 x}{2LEI}(L - x)^2 \quad \leftarrow$$

$$v' = -\frac{M_0}{2LEI}(L - x)(L - 3x)$$

MAXIMUM DEFLECTION

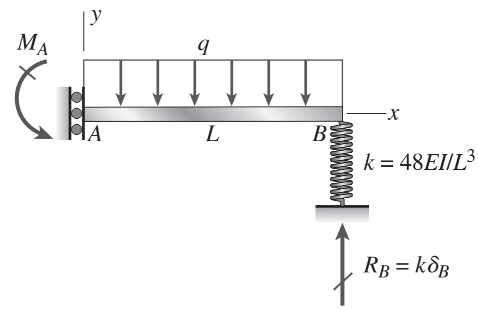
Set $v' = 0$ and solve for x:

$$x_1 = L \text{ and } x_2 = \frac{L}{3}$$

Maximum deflection occurs at $x_2 = \frac{L}{3}$.

$$\delta_{\max} = -v\left(\frac{L}{3}\right) = \frac{2M_0 L^2}{27EI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.4-4 A beam with a uniform load has a guided support at one end and spring support at the other. The spring has stiffness $k = 48EI/L^3$. Derive the equation of the deflection curve by starting with the third-order differential equation (the shear-force equation). Also, determine the angle of rotation θ_B at support B.



Solution 9.4-4

SHEAR-FORCE EQUATION

$$EIv''' = V = -qx$$

$$EIv'' = -\frac{qx^2}{2} + C_1$$

B.C. $v''(L) = M(L) = 0 \qquad C_1 = \frac{qL^2}{2}$

$$EIv'' = \frac{qL^2}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qL^2x}{2} - \frac{qx^3}{6} + C_2$$

$$EIv = \frac{qL^2x^2}{4} - \frac{qx^4}{24} + C_2x + C_3$$

B.C. $v'(0) = 0 \qquad C_2 = 0$

B.C. $v(L) = \frac{qL}{k} = -\frac{qL^4}{48EI}$

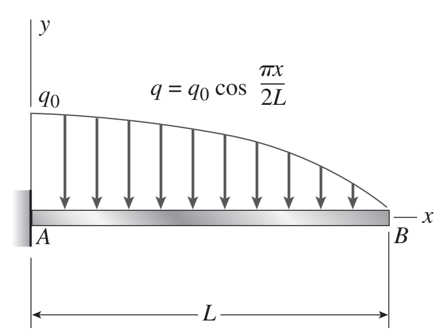
$$C_3 = -\frac{11qL^4}{48}$$

$$v(x) = -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4) \qquad \leftarrow$$

$$\theta_B = -v'(L) = -\frac{qL^3}{3EI} \quad \text{(Counterclockwise)} \qquad \leftarrow$$

Problem 9.4-5 The distributed load acting on a cantilever beam AB has an intensity q given by the expression $q_0 \cos \pi x/2L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-5 Cantilever beam (cosine load)

LOAD EQUATION (EQ. 9-12c)

$$EIv'''' = -q = -q_0 \cos \frac{\pi x}{2L}$$

$$EIv''' = -q_0\left(\frac{2L}{\pi}\right) \sin \frac{\pi x}{2L} + C_1$$

B.C. 1 $EIv''' = V$ $\quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0 L}{\pi}$

$$EIv'' = q_0\left(\frac{2L}{\pi}\right)^2 \cos \frac{\pi x}{2L} + \frac{2q_0 Lx}{\pi} + C_2$$

B.C. 2 $EIv'' = M$ $\quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{2q_0 L^2}{\pi}$

$$EIv' = q_0\left(\frac{2L}{\pi}\right)^3 \sin \frac{\pi x}{2L} + \frac{q_0 Lx^2}{\pi} - \frac{2q_0 L^2 x}{\pi} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -q_0\left(\frac{2L}{\pi}\right)^4 \cos \frac{\pi x}{2L} + \frac{q_0 Lx^3}{3\pi} - \frac{q_0 L^2 x^2}{\pi} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = \frac{16 q_0 L^4}{\pi^4}$

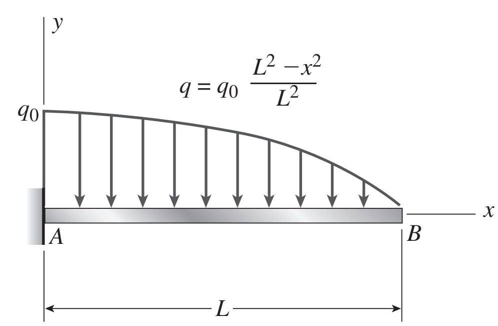
$$v = -\frac{q_0 L}{3\pi^4 EI}\left(48L^3 \cos \frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3\right) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{2q_0 L^4}{3\pi^4 EI}(\pi^3 - 24) \quad \leftarrow$$

(These results agree with Case 10, Table G-1.)

Problem 9.4-6 A cantilever beam AB is subjected to a parabolically varying load of intensity $q = q_0(L^2 - x^2)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B and angle of rotation θ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-6 Cantilever beam (parabolic load)

LOAD EQUATION (EQ. 9-12c)

$$EIv'''' = -q = -\frac{q_0}{L^2}(L^2 - x^2)$$

$$EIv''' = -\frac{q_0}{L^2}\left(L^2 x - \frac{x^3}{3}\right) + C_1$$

B.C. 1 $EIv''' = V$ $\quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0 L}{3}$

$$EIv'' = -\frac{q_0}{L^2}\left(\frac{L^2 x^2}{2} - \frac{x^4}{12}\right) + \frac{2q_0 L}{3}x + C_2$$

B.C. 2 $EIv'' = M$ $\quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{q_0 L^2}{4}$

$$EIv' = -\frac{q_0}{L^2}\left(\frac{L^2 x^3}{6} - \frac{x^5}{60}\right) + \frac{q_0 Lx^2}{3} - \frac{q_0 L^2 x}{4} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{q_0}{L^2}\left(\frac{L^2 x^4}{24} - \frac{x^6}{360}\right) + \frac{q_0 Lx^3}{9} - \frac{q_0 L^2 x^2}{8} + C_4$$

B.C. 4 $\nu(0) = 0 \qquad \therefore C_4 = 0$

$$\nu = -\frac{q_0 x^2}{360\,L^2 EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4) \qquad \leftarrow$$

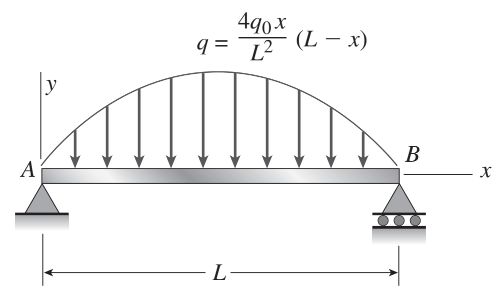
$$\delta_B = -\nu(L) = \frac{19q_0L^4}{360\,EI} \qquad \leftarrow$$

$$\nu' = -\frac{q_0 x}{60L^2EI}(15L^4 - 20L^3x + 10L^2x^2 - x^4)$$

$$\theta_B = -\nu'(L) = \frac{q_0L^3}{15EI} \qquad \leftarrow$$

Problem 9.4-7 A beam on simple supports is subjected to a parabolically distributed load of intensity $q = 4q_0x(L - x)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{max}. Use the fourthorder differential equation of the deflection curve (the load equation).



Solution 9.4-7 Simple beam (parabolic load)

LOAD EQUATION (EQ. 9-12c)

$$EI\nu'''' = -q = -\frac{4q_0x}{L^2}(L - x) = -\frac{4q_0}{L^2}(Lx - x^2)$$

$$EI\nu''' = -\frac{2q_0}{3L^2}(3Lx^2 - 2x^3) + C_1$$

$$EI\nu'' = -\frac{q_0}{3L^2}(2Lx^3 - x^4) + C_1x + C_2$$

B.C. 1 $EI\nu'' = M \qquad EI\nu''(0) = 0 \qquad \therefore C_2 = 0$

B.C. 2 $EI\nu''(L) = 0 \qquad \therefore C_1 = \dfrac{q_0L}{3}$

$$EI\nu' = -\frac{q_0}{30L^2}(-5L^3x^2 + 5Lx^4 - 2x^5) + C_3$$

B.C. 3 (Symmetry) $\nu'\left(\dfrac{L}{2}\right) = 0 \qquad \therefore C_3 = -\dfrac{q_oL^3}{30}$

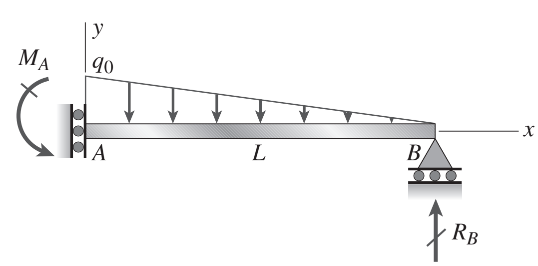
$$EI\nu = -\frac{q_0}{30L^2}\left(L^5x - \frac{5L^3x^3}{3} + Lx^5 - \frac{x^6}{3}\right) + C_4$$

B.C. 4 $\nu(0) = 0 \qquad \therefore C_4 = 0$

$$\nu = -\frac{q_0x}{90L^2EI}(3L^5 - 5L^3x^2 + 3Lx^4 - x^5) \qquad \leftarrow$$

$$\delta_{max} = -\nu\left(\frac{L}{2}\right) = \frac{61q_0L^4}{5760\,EI} \qquad \leftarrow$$

Problem 9.4-8 Derive the equation of the deflection curve for beam AB, with guided support at A and roller at B, carrying a triangularly distributed load of maximum intensity q_0 (see figure). Also, determine the maximum deflection δ_{max} of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-8

LOAD EQUATION

$$EIv'''' = -q = -q_0 + \frac{q_0 x}{L}$$

$$EIv''' = -q_0 x + \frac{q_0 x^2}{2L} + C_1$$

B.C. $v'''(0) = V(0) = 0 \qquad C_1 = 0$

$$EIv''' = -q_0 x + \frac{q_0 x^2}{2L}$$

$$EIv'' = -\frac{q_0 x^2}{2} + \frac{q_0 x^3}{6L} + C_2$$

B.C. $v''(L) = M(L) = 0 \qquad C_2 = \frac{q_0 L^2}{3}$

$$EIv'' = -\frac{q_0 x^2}{2} + \frac{q_0 x^3}{6L} + \frac{q_0 L^2}{3}$$

$$EIv' = -\frac{q_0 x^3}{6} + \frac{q_0 x^4}{24L} + \frac{q_0 L^2 x}{3} + C_3$$

$$EIv = -\frac{q_0 x^4}{24} + \frac{q_0 x^5}{120L} + \frac{q_0 L^2 x^2}{6} + C_3 x + C_4$$

B.C. $v'(0) = 0 \qquad C_3 = 0$

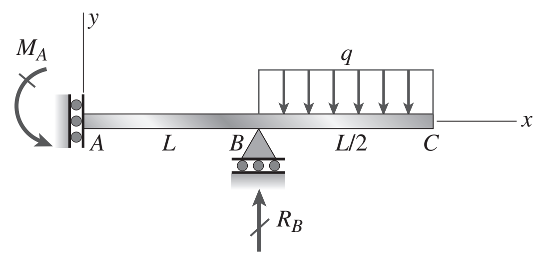
B.C. $v(L) = 0 \qquad C_4 = -\frac{2q_0 L^4}{15}$

$$v(x) = \frac{q_0}{120EIL}(-5x^4 L + x^5 + 20L^3 x^2 - 16L^5) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$\delta_{max} = -v(0) = \frac{2q_0 L^4}{15EI} \quad \leftarrow$$

Problem 9.4-9 Derive the equations of the deflection curve for beam ABC, with guided support at A and roller support at B, supporting a uniform load of intensity q acting on the over-hang portion of the beam (see figure). Also, determine deflection δ_C and angle of rotation θ_C. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-9

LOAD EQUATION

$$EIv'''' = -q = 0 \quad (0 \le x \le L)$$

$$EIv''' = C_1 \quad (0 \le x \le L)$$

$$EIv'' = C_1 x + C_2 \quad (0 \le x \le L)$$

B.C. $v'''(0) = V(0) = 0 \qquad C_1 = 0$

B.C. $v''(0) = M(0) = -\frac{qL^2}{8} \qquad C_2 = -\frac{qL^2}{8}$

$$EIv'' = -\frac{qL^2}{8}$$

$$EIv' = -\frac{qL^2 x}{8} + C_3$$

B.C. $v'(0) = 0 \qquad C_3 = 0$

$$EIv = -\frac{qL^2 x^2}{16} + C_4$$

B.C. $v(L) = 0 \qquad C_4 = \frac{qL^4}{16}$

$$v(x) = -\frac{qL^2}{16EI}(x^2 - L^2) \quad (0 \le x \le L) \quad \leftarrow$$

LOAD EQUATION

$$EIv'''' = -q \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$EIv''' = -qx + C_5 \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$EIv'' = \frac{-qx^2}{2} + C_5x + C_6 \quad \left(L \le x \le \frac{3L}{2}\right)$$

$$\text{B.C. } v'''\left(\frac{3L}{2}\right) = V\left(\frac{3L}{2}\right) = 0 \qquad C_5 = \frac{3qL}{2}$$

$$\text{B.C. } v''\left(\frac{3L}{2}\right) = M\left(\frac{3L}{2}\right) = 0 \qquad C_6 = \frac{9qL^2}{8}$$

$$EIv'' = \frac{-qx^2}{2} + \frac{3qLx}{2} - \frac{9qL^2}{8}$$

$$EIv' = \frac{-qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + C_7$$

$$\text{B.C. } v'_L(L) = v'_R(L)$$

$$-\frac{qL^3}{8} = \frac{-qL^3}{6} + \frac{3qL^3}{4} - \frac{9qL^3}{8} + C_7$$

$$C_7 = \frac{5}{12}qL^3$$

$$EIv' = \frac{-qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + \frac{5}{12}qL^3$$

$$EIv = \frac{-qx^4}{24} + \frac{3qLx^3}{12} - \frac{9qL^2x^2}{16} + \frac{5}{12}qL^3x + C_8$$

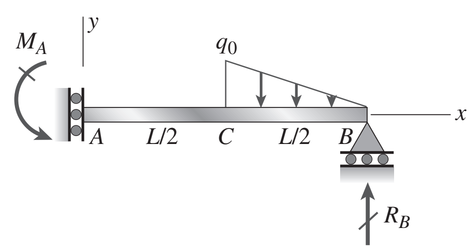
$$\text{B.C. } v(L) = 0 \qquad C_8 = \frac{-1}{16}qL^4$$

$$v(x) = \frac{-q}{48EI}(-20xL^3 + 27L^2x^2 - 12Lx^3 + 2x^4 + 3L^4) \quad \left(L \le x \le \frac{3L}{2}\right) \quad \leftarrow$$

$$\delta_C = -v\left(\frac{3L}{2}\right) = \frac{9qL^4}{128EI} \quad \leftarrow$$

$$\theta_C = -v'\left(\frac{3L}{2}\right) = \frac{7qL^3}{48EI} \quad \text{(Clockwise)} \quad \leftarrow$$

Problem 9.4-10 Derive the equations of the deflection curve for beam AB, with guided support at A and roller support at B, supporting a distributed load of maximum intensity q_0 acting on the right-hand half of the beam (see figure). Also, determine deflection δ_A, angle of rotation θ_B, and deflection δ_C at the midpoint. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-10

LOAD EQUATION

$$EIv'''' = -q = 0 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv''' = C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv'' = C_1x + C_2 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\text{B.C. } v'''(0) = V(0) = 0 \qquad C_1 = 0$$

$$\text{B.C. } v''(0) = M(0) = \frac{q_0L^2}{12} \qquad C_2 = \frac{q_0L^2}{12}$$

$$EIv'' = \frac{q_0L^2}{12} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$EIv' = \frac{q_0L^2x}{12} + C_3 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. $v'(0) = 0 \qquad C_3 = 0$

$$EIv = \frac{q_0 L^2 x^2}{24} + C_4 \quad \left(0 \le x \le \frac{L}{2}\right)$$

LOAD EQUATION

$$EIv'''' = -q_0 + \frac{2q_0}{L}\left(x - \frac{L}{2}\right) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv'''' = -2q_0 + \frac{2q_0 x}{L}$$

$$EIv''' = -2q_0 x + \frac{q_0 x^2}{L} + C_5 \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv'' = -q_0 x^2 + \frac{q_0 x^3}{3L} + C_5 x + C_6 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. $v'''\left(\frac{L}{2}\right) = V\left(\frac{L}{2}\right) = 0 \qquad C_5 = \frac{3q_0 L}{4}$

B.C. $v''\left(\frac{L}{2}\right) = M\left(\frac{L}{2}\right) = \frac{q_0 L^2}{12} \qquad C_6 = \frac{q_0 L^2}{12}$

$$EIv'' = -q_0 x^2 + \frac{q_0 x^3}{3L} + \frac{3q_0 L x}{4} - \frac{q_0 L^2}{12}$$

$$EIv' = -\frac{q_0 x^3}{3} + \frac{q_0 x^4}{12L} + \frac{3q_0 L x^2}{8} - \frac{q_0 L^2 x}{12} + C_7$$

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right) \qquad C_7 = \frac{5}{192} q_0 L^3$

$$EIv' = -\frac{q_0 x^3}{3} + \frac{q_0 x^4}{12L} + \frac{3q_0 L x^2}{8} + \frac{q_0 L^2 x}{12} + \frac{5q_0 L^3}{192}$$

$$EIv = -\frac{q_0 x^4}{12} + \frac{q_0 x^5}{60L} + \frac{q_0 L x^3}{8} - \frac{q_0 L^2 x^2}{24} + \frac{5q_0 L^3 x}{192} + C_8$$

B.C. $v(L) = 0 \qquad C_8 = \frac{-41}{960} q_0 L^4$

$$EIv = -\frac{q_0 x^4}{12} + \frac{q_0 x^5}{60L} + \frac{q_0 L x^3}{8} - \frac{q_0 L^2 x^2}{24} + \frac{5q_0 L^3 x}{192} - \frac{41}{960} q_0 L^4 \quad \left(\frac{L}{2} \le x \le L\right) \quad \leftarrow$$

B.C. $v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$

$$\frac{q_0 L^2 \left(\frac{L}{2}\right)^2}{24} + C_4 = -\frac{q_0 \left(\frac{L}{2}\right)^4}{12} + \frac{q_0 \left(\frac{L}{2}\right)^5}{60L} + \frac{q_0 L \left(\frac{L}{2}\right)^3}{8} - \frac{q_0 L^2 \left(\frac{L}{2}\right)^2}{24} + \frac{5q_0 L^3 \frac{L}{2}}{192} - \frac{41}{960} q_0 L^4$$

$$C_4 = \frac{-19}{480} q_0 L^4$$

$$EIv = \frac{q_0 L^2 x^2}{24} - \frac{19}{480} q_0 L^4 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$v(x) = -\frac{q_0 L^2}{480EI}(-20x^2 + 19L^2) \quad \left(0 \le x \le L\right) \quad \leftarrow$$

$$v(x) = -\frac{q_0}{960LEI}(80x^4 L - 16x^5 - 120L^2 x^3 + 40L^3 x^2 - 25L^4 x + 41L^5) \quad \left(\frac{L}{2} \le x \le L\right) \quad \leftarrow$$

$$\delta_A = -v(0) = \frac{19}{480EI} q_0 L^4 \quad \leftarrow$$

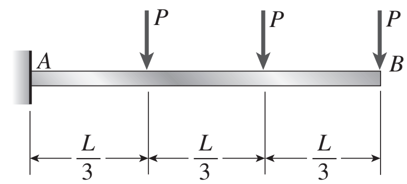
$$\theta_B = -v'(L) = -\frac{13}{192EI} q_0 L^3 \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{7}{240EI} q_0 L^4 \quad \leftarrow$$

Method of Superposition

The problems for Section 9.5 are to be solved by the method of superposition. All beams have constant flexural rigidity EI.

Problem 9.5-1 A cantilever beam AB carries three equally spaced concentrated loads, as shown in the figure. Obtain formulas for the angle of rotation θ_B and deflection δ_B at the free end of the beam.



Solution 9.5-1 Cantilever beam with 3 loads

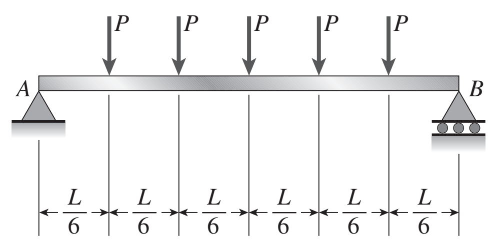
Table G-1, Cases 4 and 5

$$\theta_B = \frac{P\left(\frac{L}{3}\right)^2}{2EI} + \frac{P\left(\frac{2L}{3}\right)^2}{2EI} + \frac{PL^2}{2EI} = \frac{7PL^2}{9EI} \quad \leftarrow$$

$$\delta_B = \frac{P\left(\frac{L}{3}\right)^2}{6EI}\left(3L - \frac{L}{3}\right) + \frac{P\left(\frac{2L}{3}\right)^2}{6EI}\left(3L - \frac{2L}{3}\right) + \frac{PL^3}{3EI} = \frac{5PL^3}{9EI} \quad \leftarrow$$

Problem 9.5-2 A simple beam AB supports five equally spaced loads P (see figure).

(a) Determine the deflection δ_1 at the midpoint of the beam.
(b) If the same total load ($5P$) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?
(c) Calculate the ratio of δ_1 to δ_2.



Solution 9.5-2 Simple beam with 5 loads

(a) Table G-2, Cases 4 and 6

$$\delta_1 = \frac{P\left(\frac{L}{6}\right)}{24\,EI}\left[3L^2 - 4\left(\frac{L}{6}\right)^2\right] + \frac{P\left(\frac{L}{3}\right)}{24EI}\left[3L^2 - 4\left(\frac{L}{3}\right)^2\right] + \frac{PL^3}{48EI} = \frac{11PL^3}{144EI} \quad \leftarrow$$

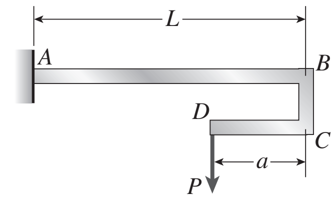
(b) Table G-2, Case 1 $qL = 5P$

$$\delta_2 = \frac{5qL^4}{384EI} = \frac{25PL^3}{384EI} \quad \leftarrow$$

(c) $$\frac{\delta_1}{\delta_2} = \frac{11}{144}\left(\frac{384}{25}\right) = \frac{88}{75} = 1.173 \quad \leftarrow$$

Problem 9.5-3 The cantilever beam *AB* shown in the figure has an extension *BCD* attached to its free end. A force *P* acts at the end of the extension.

(a) Find the ratio *a/L* so that the vertical deflection of point *B* will be zero.
(b) Find the ratio *a/L* so that the angle of rotation at point *B* will be zero.



Solution 9.5-3 Cantilever beam with extension

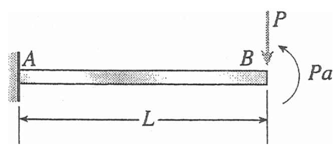


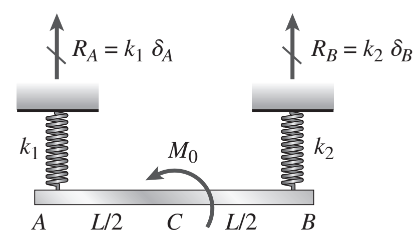
Table G-1, Cases 4 and 6

(a) $\delta_B = \dfrac{PL^3}{3EI} - \dfrac{PaL^2}{2EI} = 0 \qquad \dfrac{a}{L} = \dfrac{2}{3}$ ←

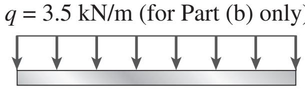
(b) $\theta_B = \dfrac{PL^2}{2EI} - \dfrac{PaL}{EI} = 0 \qquad \dfrac{a}{L} = \dfrac{1}{2}$ ←

Problem 9.5-4 Beam *ACB* hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity *EI*.

(a) What is the downward displacement of point *C*, which is at the midpoint of the beam, when the moment M_0 is applied? Data for the structure are as follows: $M_0 = 10.0$ kN·m, $L = 1.8$ m, $EI = 216$ kN·m^2, $k_1 = 250$ kN/m, and $k_2 = 160$ kN/m.
(b) Repeat (a) but remove M_0 and apply uniform load $q = 3.5$ kN/m to the entire beam.



$q = 3.5$ kN/m (for Part (b) only)



Solution 9.5-4

$M_0 = 10.0\text{ kN}\cdot\text{m} \qquad L = 1.8\text{ m} \qquad EI = 216\text{ kN}\cdot\text{m}^2$

$k_1 = 250\text{ kN/m} \qquad k_2 = 160\text{ kN/m}$

$q = 3.5\text{ kN/m}$

(a) $R_A = \dfrac{M_0}{L} \qquad R_B = -\dfrac{M_0}{L}$

$\delta_A = \dfrac{R_A}{k_1} \qquad \delta_B = \dfrac{R_B}{k_2}$

$\delta_A = 22.22$ mm Downward

$\delta_B = -34.72$ mm Upward

Table G-2, Case 8

$\delta_C = 0 + \dfrac{1}{2}\cdot(\delta_A + \delta_B)$

$\delta_C = -6.25$ mm Upward ←

(b) $R_A = \dfrac{qL}{2} \qquad R_B = R_A$

$\delta_A = \dfrac{R_A}{k_1} \qquad \delta_B = \dfrac{R_B}{k_2}$

$\delta_A = 12.60$ mm

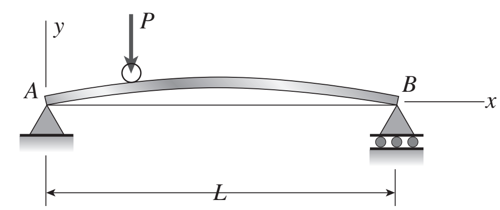
$\delta_B = 19.69$ mm

Table G-2, Case 1

$$\delta_C = \frac{5qL^4}{384EI} + \frac{1}{2}(\delta_A + \delta_B)$$

$\delta_C = 18.36$ mm Downward $\leftarrow$

Problem 9.5-5 What must be the equation $y = f(x)$ of the axis of the slightly curved beam AB (see figure) *before* the load is applied in order that the load P, moving along the bar, always stays at the same level?



Solution 9.5-5 Slightly curved beam

Let x = distance to load P

δ = downward deflection at load P

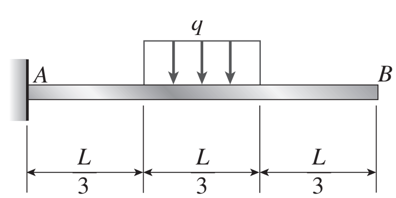
Table G-2, Case 5:

$$\delta = \frac{P(L - x)x}{6LEI}[L^2 - (L - x)^2 - x^2] = \frac{Px^2(L - x)^2}{3LEI}$$

Initial upward displacement of the beam must equal δ.

$$\therefore\ y = \frac{Px^2(L - x)^2}{3LEI} \qquad \leftarrow$$

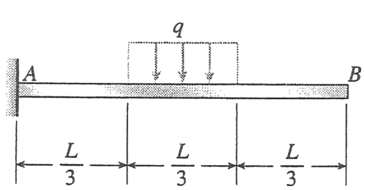
Problem 9.5-6 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB having a uniform load of intensity q acting over the middle third of its length (see figure).



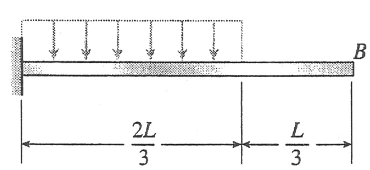
Solution 9.5-6 Cantilever beam (partial uniform load)

q = intensity of uniform load

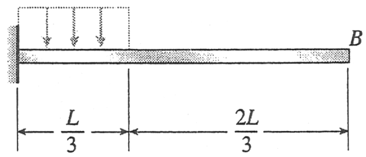
Original load on the beam:



Load No. 1:



Load No. 2:



SUPERPOSITION:

Original load = Load No. 1 minus Load No. 2

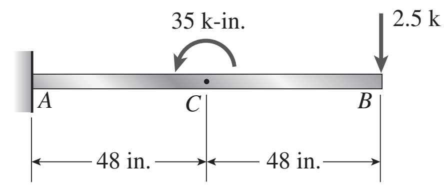
Table G-1, Case 2

$$\theta_B = \frac{q}{6EI}\left(\frac{2L}{3}\right)^3 - \frac{q}{6EI}\left(\frac{L}{3}\right)^3 = \frac{7qL^3}{162EI} \quad \leftarrow$$

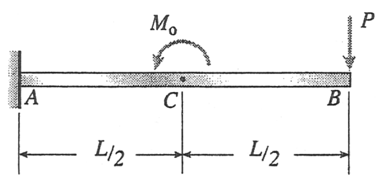
$$\delta_B = \frac{q}{24EI}\left(\frac{2L}{3}\right)^3\left(4L - \frac{2L}{3}\right) - \frac{q}{24EI}\left(\frac{L}{3}\right)^3\left(4L - \frac{L}{3}\right)$$

$$= \frac{23qL^4}{648EI} \quad \leftarrow$$

Problem 9.5-7 The cantilever beam ACB shown in the figure has flexural rigidity $EI = 2.1 \times 10^6$ k-in.^{2} Calculate the downward deflections δ_C and δ_B at points C and B, respectively, due to the simultaneous action of the moment of 35 k-in. applied at point C and the concentrated load of 2.5 k applied at the free end B.



Solution 9.5-7 Cantilever beam (two loads)



$EI = 2.1 \times 10^6$ k-in.^{2}

$M_0 = 35$ k-in.

$P = 2.5$ k

$L = 96$ in.

Table G-1, Cases 4, 6, and 7

$$\delta_C = -\frac{M_0(L/2)^2}{2EI} + \frac{P(L/2)^2}{6EI}\left(3L - \frac{L}{2}\right)$$

$$= -\frac{M_0L^2}{8EI} + \frac{5PL^3}{48EI} \quad (+ = \text{downward deflection})$$

$$\delta_B = -\frac{M_0(L/2)}{2EI}\left(2L - \frac{L}{2}\right) + \frac{PL^3}{3EI}$$

$$= -\frac{3M_0L^2}{8EI} + \frac{PL^3}{3EI} \quad (+ = \text{downward deflection})$$

SUBSTITUTE NUMERICAL VALUES:

$\delta_C = -0.01920$ in. $+ 0.10971$ in.

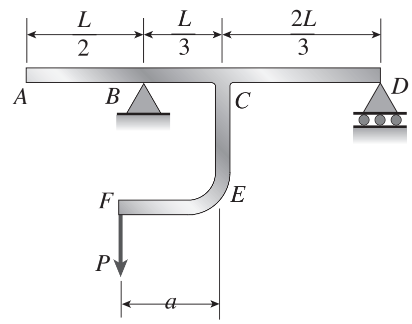
$= 0.0905$ in. $\leftarrow$

$\delta_B = -0.05760$ in. $+ 0.35109$ in.

$= 0.293$ in. $\leftarrow$

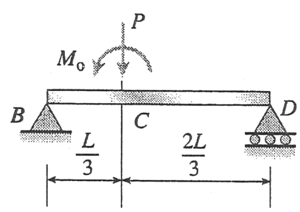
Problem 9.5-8 A beam *ABCD* consisting of a simple span *BD* and an overhang *AB* is loaded by a force *P* acting at the end of the bracket *CEF* (see figure).

(a) Determine the deflection δ_A at the end of the overhang.
(b) Under what conditions is this deflection upward? Under what conditions is it downward?



Solution 9.5-8 Beam with bracket and overhang

Consider part *BD* of the beam.



$M_0 = Pa$

Table G-2, Cases 5 and 9

$$\theta_B = \frac{P(L/3)(2L/3)(5L/3)}{6LEI} + \frac{Pa}{6LEI}\left[6\left(\frac{L^2}{3}\right) - 3\left(\frac{L^2}{9}\right) - 2L^2\right]$$

$$= \frac{PL}{162EI}(10L - 9a) \quad (+ = \text{clockwise angle})$$

(a) Deflection at the end of the overhang

$$\delta_A = \theta_B\left(\frac{L}{2}\right) = \frac{PL^2}{324\,EI}(10L - 9a) \quad \leftarrow$$

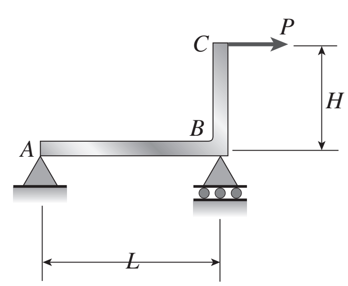
$(+ = \text{upward deflection})$

(b) Deflection is upward when $\frac{a}{L} < \frac{10}{9}$ and downward when $\frac{a}{L} > \frac{10}{9}$ $\leftarrow$

Problem 9.5-9 A horizontal load *P* acts at end *C* of the bracket *ABC* shown in the figure.

(a) Determine the deflection δ_C of point *C*.
(b) Determine the maximum upward deflection δ_{max} of member *AB*.

Note: Assume that the flexural rigidity *EI* is constant throughout the frame. Also, disregard the effects of axial deformations and consider only the effects of bending due to the load *P*.



Solution 9.5-9 Bracket *ABC*

BEAM *AB*

$M_0 = PH$

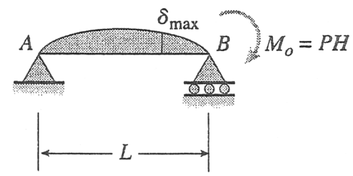


Table G-2, Case 7: $\theta_B = \dfrac{M_0 L}{3EI} = \dfrac{PHL}{3EI}$

(a) ARM *BC* Table G-1, Case 4

$$\delta_C = \frac{PH^3}{3EI} + \theta_B H = \frac{PH^3}{3EI} + \frac{PH^2 L}{3EI}$$

$$= \frac{PH^2}{3EI}(L + H) \quad \leftarrow$$

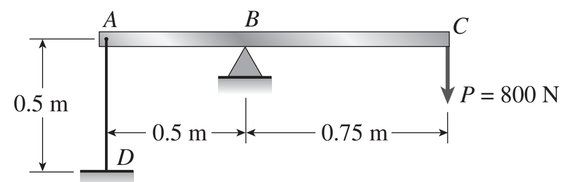
(b) MAXIMUM DEFLECTION OF BEAM *AB*

Table G-2,

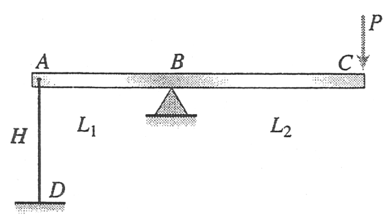
Case 7: $\delta_{max} = \dfrac{M_0 L^2}{9\sqrt{3}EI} = \dfrac{PHL^2}{9\sqrt{3}EI} \quad \leftarrow$

Problem 9.5-10 A beam *ABC* having flexural rigidity $EI = 75\ \text{kN}\cdot\text{m}^2$ is loaded by a force $P = 800$ N at end *C* and tied down at end *A* by a wire having axial rigidity $EA = 900$ kN (see figure).

What is the deflection at point *C* when the load *P* is applied?



Solution 9.5-10 Beam tied down by a wire



$EI = 75\ \text{kN}\cdot\text{m}^2$

$P = 800$ N

$EA = 900$ kN

$H = 0.5$ m $\quad L_1 = 0.5$ m

$L_2 = 0.75$ m

CONSIDER *BC* AS A CANTILEVER BEAM

Table G-1, Case 4: $\delta'_C = \dfrac{PL_2^3}{3EI}$

CONSIDER *AB* AS A SIMPLE BEAM

$M_0 = PL_2$

Table G-2, Case 7: $\theta'_B = \dfrac{M_0 L_1}{3EI} = \dfrac{PL_1 L_2}{3EI}$

CONSIDER THE STRETCHING OF WIRE *AD*

$$\delta'_A = (\text{Force in } AD)\left(\frac{H}{EA}\right) = \left(\frac{PL_2}{L_1}\right)\left(\frac{H}{EA}\right) = \frac{PL_2 H}{EAL_1}$$

DEFLECTION δ_C OF POINT *C*

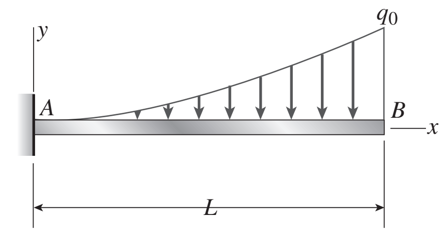
$$\delta_c = \delta'_c + \theta'_B(L_2) + \delta'_B\left(\frac{L_2}{L_1}\right)$$

$$= \frac{PL_2^3}{3EI} + \frac{PL_1 L_2^2}{3EI} + \frac{PL_2^2 H}{EAL_1^2} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$\delta_C = 1.50\text{ mm} + 1.00\text{ mm} + 1.00\text{ mm} = 3.50\text{ mm} \quad \leftarrow$

Problem 9.5-11 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB supporting a parabolic load defined by the equation $q = q_0x^2/L^2$ (see figure).



Solution 9.5-11 Cantilever beam (parabolic load)

LOAD: $q = \dfrac{q_0x^2}{L^2}$ $\quad qdx$ = element of load

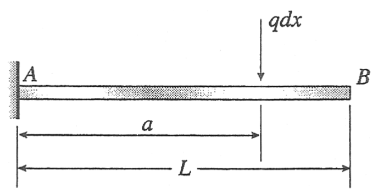


TABLE G-1, CASE 5 (Set a equal to x)

$$\theta_B = \int_0^L \frac{(qdx)(x^2)}{2EI} = \frac{1}{2EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right) x^2dx$$

$$= \frac{q_0}{2EIL^2}\int_0^L x^4dx = \frac{q_0L^3}{10EI} \quad \longleftarrow$$

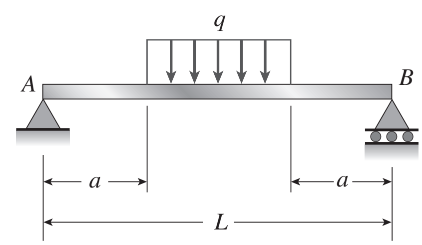
$$\delta_B = \int_0^L \frac{(qdx)(x^2)}{6EI}(3L - x)$$

$$= \frac{1}{6EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right)(x^2)(3L - x)\,dx$$

$$= \frac{q_0}{6EIL^2}\int_0^L (x^4)(3L - x)\,dx = \frac{13q_0L^4}{180EI} \quad \longleftarrow$$

Problem 9.5-12 A simple beam AB supports a uniform load of intensity q acting over the middle region of the span (see figure).

Determine the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.



Solution 9.5-12 Simple beam (partial uniform load)

LOAD: qdx = element of load

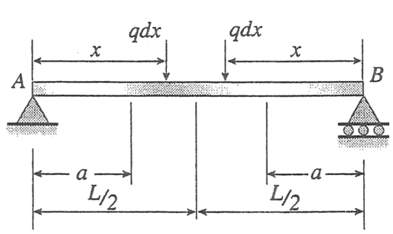


TABLE G-2, CASE 6 $\quad \theta_A = \dfrac{Pa(L - a)}{2EI}$

Replace P by qdx Replace a by x

Integrate x from a to $L/2$

$$\theta_A = \int_a^{L/3} \frac{qdx}{2EI}(x)(L - x) = \frac{q}{2EI}\int_a^{L/2} (xL - x^2)dx$$

$$= \frac{q}{24EI}(L^3 - 6a^2L + 4a^3) \quad \longleftarrow$$

TABLE G-2, CASE 6 $\quad \delta_{max} = \dfrac{Pa}{24EI}(3L^2 - 4a^2)$

Replace P by qdx Replace a by x

Integrate x from a to $L/2$

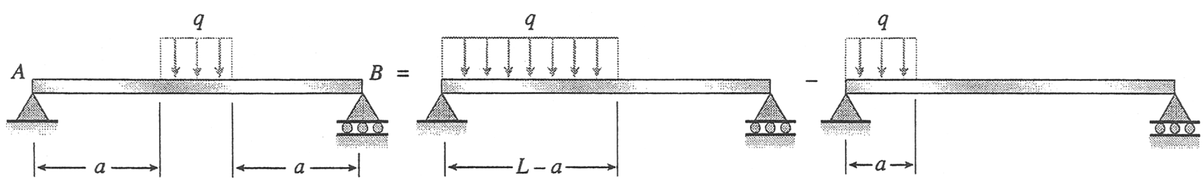
$$\delta_{max} = \int_a^{L/2} \frac{q\,dx}{24EI}(x)(3L^2 - 4x^2)$$
$$= \frac{q}{24EI}\int_a^{L/2}(3L^2x - 4x^3)dx$$
$$= \frac{q}{384EI}(5L^4 - 24a^2L^2 + 16a^4) \quad \leftarrow$$

ALTERNATE SOLUTION (not recommended; algebra is extremely lengthy)

Table G-2, Case 3

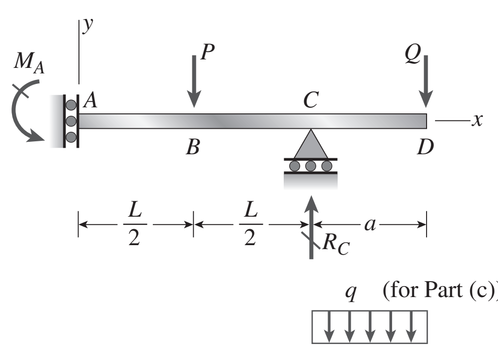
$$\theta_A = \frac{q(L-a)^2}{24LEI}[2L - (L-a)]^2 - \frac{qa^2}{24LEI}(2L-a)^2$$
$$= \frac{q}{24EI}(L^3 - 6La^2 + 4a^3) \quad \leftarrow$$

$$\delta_{max} = \frac{q(L/2)}{24LEI}\Big[(L-a)^4 - 4L(L-a)^3 + 4L^2(L-a)^2 + 2(L-a)^2\left(\frac{L}{2}\right)^2 - 4L(L-a)\left(\frac{L}{2}\right)^2 + L\left(\frac{L}{2}\right)^3\Big]$$
$$= \frac{qa^2}{24LEI}\Big[-La^2 + 4L^2\left(\frac{L}{2}\right) + a^2\left(\frac{L}{2}\right) - 6L\left(\frac{L}{2}\right)^2 + 2\left(\frac{L}{2}\right)^3\Big]$$
$$\delta_{max} = \frac{q}{384EI}(5L^4 - 24L^2a^2 + 16a^4) \quad \leftarrow$$



Problem 9.5-13 The overhanging beam *ABCD* supports two concentrated loads *P* and *Q* (see figure).

(a) For what ratio *P/Q* will the deflection at point *B* be zero?
(b) For what ratio will the deflection at point *D* be zero?
(c) If *Q* is replaced by uniform load with intensity *q* (on the overhang), repeat (a) and (b) but find ratio *P/(qa)*



Solution 9.5-13

(a) DEFLECTION AT POINT *B*

Table G-2 Cases 6 and 10

$$\delta_B = \frac{P\left(\frac{L}{2}\right)}{6EI}\left[3\left(\frac{L}{2}\right)(2L) - 3\left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2\right] - \frac{Qa\left(\frac{L}{2}\right)}{2EI}\left[(2L) - \left(\frac{L}{2}\right)\right]$$

$$\delta_B = 0 \qquad \frac{P}{Q} = \frac{9a}{4L} \quad \leftarrow$$

(b) DEFLECTION AT POINT D

Table G-2 Case 6; Table G-1 Case 4; Table G-2 Case 10

$$\delta_D = -\frac{P\left(\frac{L}{2}\right)\left[(2L) - \frac{L}{2}\right]}{2EI}(a) + \frac{Qa^3}{3EI} + \frac{Qa(2L)}{2EI}(a)$$

$$\delta_D = 0 \qquad \frac{P}{Q} = \frac{8a(3L + a)}{9L^2} \qquad \leftarrow$$

(c.1) DEFLECTION AT POINT B

Table G-2 Cases 6 and 10

$$\delta_B = \frac{P\left(\frac{L}{2}\right)}{6EI}\left[3\left(\frac{L}{2}\right)(2L) - 3\left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2\right] - \frac{\left(\frac{qa^2}{2}\right)\left(\frac{L}{2}\right)}{2EI}\left[(2L) - \left(\frac{L}{2}\right)\right]$$

$$\delta_B = 0 \qquad \frac{P}{qa} = \frac{9a}{8L} \qquad \leftarrow$$

(c.2) DEFLECTION AT POINT D

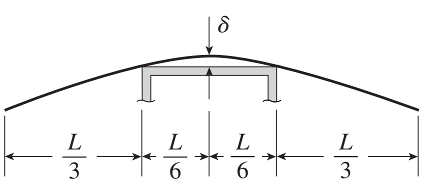
Table G-2 Case 6; Table G-1 Case 1; Table G-2 Case 10

$$\delta_D = -\frac{P\left(\frac{L}{2}\right)\left[(2L) - \frac{L}{2}\right]}{2EI}(a) + \frac{qa^4}{8EI} + \frac{\left(\frac{qa^2}{2}\right)(2L)}{2EI}(a)$$

$$\delta_D = 0 \qquad \frac{P}{qa} = \frac{a(4L + a)}{3L^2} \qquad \leftarrow$$

Problem 9.5-14 A thin metal strip of total weight W and length L is placed across the top of a flat table of width $L/3$ as shown in the figure.

What is the clearance δ between the strip and the middle of the table? (The strip of metal has flexural rigidity EI.)



Solution 9.5-14 Thin metal strip

W = total weight $\quad q = \frac{W}{L}$

EI = flexural rigidity

FREE BODY DIAGRAM (the part of the strip above the table)

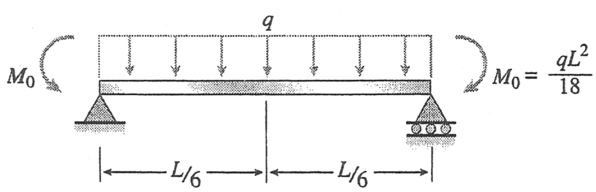


TABLE G-2, CASES 1 AND 10

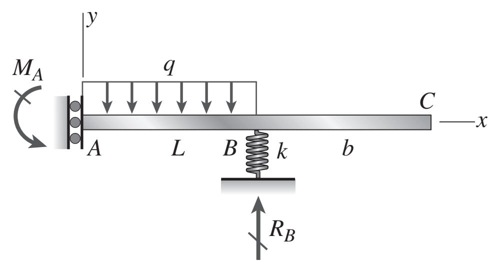
$$\delta = -\frac{5q}{384EI}\left(\frac{L}{3}\right)^4 + \frac{M_0}{8EI}\left(\frac{L}{3}\right)^2$$

$$= -\frac{5qL^4}{31{,}104EI} + \frac{qL^4}{1296EI}$$

$$= \frac{19qL^4}{31{,}104EI}$$

But $q = \frac{W}{L}$: $\quad \therefore \delta = \frac{19WL^3}{31{,}104EI} \qquad \leftarrow$

Problem 9.5-15 An overhanging beam *ABC* with flexural rigidity $EI = 15$ k-in.^{2} is supported by a guided support at *A* and by a spring of stiffness *k* at point *B* (see figure). Span *AB* has length $L = 30$ in. and carries a uniform load. The over-hang *BC* has length $b = 15$ in. For what stiffness *k* of the spring will the uniform load produce no deflection at the free end *C*?



Solution 9.5-15

$EI = 15\text{kip} \cdot \text{in}^2.$ $\quad L = 30$ in. $\quad b = 15$ in.

$R_B = qL$

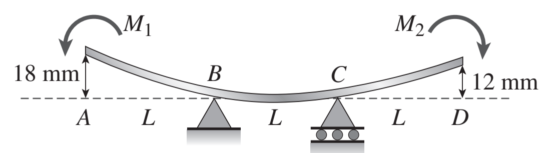
Table G-2, Case 1

$$\delta_C = \theta_B b - \delta_B = \frac{q(2L)^3}{24EI}(b) - \frac{qL}{k}$$

for $\delta_C = 0$ $\quad k = \dfrac{3EI}{bL^2}$

Therefore $\quad k = 3.33\text{lb/in} \quad \leftarrow$

Problem 9.5-16 A beam *ABCD* rests on simple supports at *B* and *C* (see figure). The beam has a slight initial curvature so that end *A* is 18 mm above the elevation of the supports and end *D* is 12 mm above. What moments M_1 and M_2, acting at points *A* and *D*, respectively, will move points *A* and *D* downward to the level of the supports? (The flexural rigidity *EI* of the beam is 2.5×10^6 N·m^2 and $L = 2.5$ m).



Solution 9.5-16

$EI = 2.5 \times 10^6 \text{ N} \cdot \text{m}^2$ $\quad \boldsymbol{L} = 2.5$ m $\quad \delta_A = 18$ mm

$\delta_D = 12$ mm

Table G-2, Case 7

$$\theta_B = \frac{M_1 L}{3EI} + \frac{M_2 L}{6EI} \qquad \theta_C = \frac{M_1 L}{6EI} + \frac{M_2 L}{3EI}$$

Deflection at point *A* and *D*

Table G-1, Case 6

$$\delta_A = \frac{M_1 L^2}{2EI} + \theta_B L \qquad \delta_D = \frac{M_2 L^2}{2EI} + \theta_C L$$

$$\delta_A = \frac{M_1 L^2}{2EI} + \left(\frac{M_1 L}{3EI} + \frac{M_2 L}{6EI}\right) L$$

$$\delta_D = \frac{M_2 L^2}{2EI} + \left(\frac{M_1 L}{6EI} + \frac{M_2 L}{3EI}\right) L$$

$$5M_1 + M_2 = \frac{6\delta_A EI}{L^2} \qquad (1)$$

$$5M_2 + M_1 = \frac{6\delta_D EI}{L^2} \qquad (2)$$

Solve equation (1) and (2)

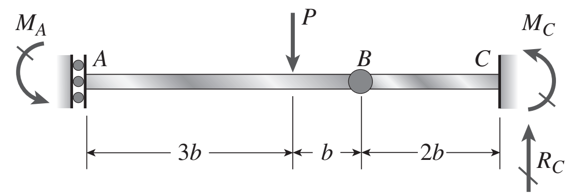
$$M_1 = \frac{EI(5\delta_A - \delta_D)}{4L^2} \qquad M_2 = \frac{EI(5\delta_D - \delta_A)}{4L^2}$$

Therefore

$M_1 = 7800 \text{ N} \cdot \text{m} \quad \leftarrow$

$M_2 = 4200 \text{ N} \cdot \text{m} \quad \leftarrow$

Problem 9.5-17 The compound beam *ABC* shown in the figure has a guided support at *A* and a fixed support at *C*. The beam consists of two members joined by a pin connection (i.e., moment release) at *B*. Find the deflection δ under the load *P*.



Solution 9.5-17

Table G-1, Case 4

$$\delta_B = \frac{P(2b)^3}{3EI}$$

Deflection under the load P

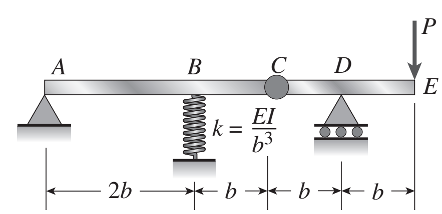
Table G-2, Case 6

$$\delta = \frac{P(b)}{6EI}\left[3(b)(8b) - 3(b)^2 - (b)^2\right] + \delta_B$$

$$\delta = \frac{P(b)}{6EI}\left[3(b)(8b) - 3b^2 - b^2\right] + \frac{P(2b)^3}{3EI}$$

$$\delta = \frac{6Pb^3}{EI} \quad \leftarrow$$

Problem 9.5-18 A compound beam *ABCDE* (see figure) consists of two parts (*ABC* and *CDE*) connected by a hinge (i.e., moment release) at *C*. The elastic support at *B* has stiffness $k = EI/b^3$ Determine the deflection δ_E at the free end *E* due to the load *P* acting at that point.



Solution 9.5-18

Consider beam ABC

$$R_B = \frac{3P}{2} \qquad \delta_B = \frac{R_B}{k} = \frac{3P}{2k} \quad \text{Upward}$$

Table G-2, Case 7; Table G-1, Case 4

$$\delta_C = \frac{Pb(2b)}{3EI}b + \frac{Pb^3}{3EI} + \delta_B\left(\frac{2b + b}{2b}\right)$$

$$= \frac{Pb(2b)}{3EI}b + \frac{Pb^3}{3EI} + \frac{3P}{2k}\left(\frac{2b + b}{2b}\right)$$

$$\delta_C = \frac{P(4b^3k + 9EI)}{4EIk} \quad \text{Upward}$$

Consider beam CDE

Table G-2, Case 7; Table G-1, Case 4

$$\delta_E = \frac{(Pb)(b)}{3EI}b + \frac{Pb^3}{3EI} + \delta_C = \frac{(Pb)(b)}{3EI}b$$

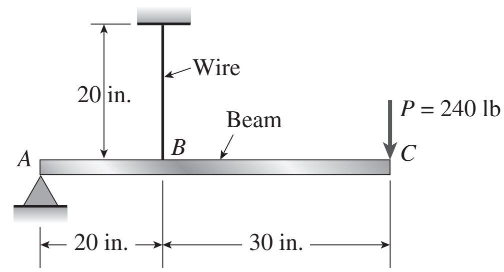
$$+ \frac{Pb^3}{3EI} + \frac{P(4b^3k + 9EI)}{4EIk}$$

for $k = \frac{EI}{b^3}$

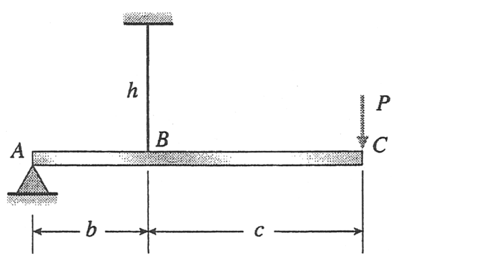
$$\delta_E = \frac{47Pb^3}{12EI} \quad \leftarrow$$

Problem 9.5-19 A stekel beam ABC is simply supported at A and held by a high-strength steel wire at B (see figure). A load $P = 240$ lb acts at the free end C. The wire has axial rigidity $EA = 1500 \times 10^3$ lb, and the beam has flexural rigidity $EI = 36 \times 10^6$ lb-in.^{2}

What is the deflection δ_C of point C due to the load P?



Solution 9.5-19 Beam supported by a wire

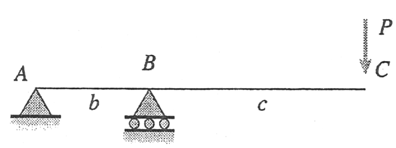


$P = 240$ lb $\quad b = 20$ in. $\quad c = 30$ in. $\quad h = 20$ in.

Beam: $EI = 36 \times 10^6$ lb-in.^{2}

Wire: $EA = 1500 \times 10^3$ lb

(1) ASSUME THAT POINT B IS ON A SIMPLE SUPPORT



$$\delta'_C = \frac{Pc^3}{3EI} + \theta'_B c = \frac{Pc^3}{3EI} + (Pc)\left(\frac{b}{3EI}\right)c$$

$$= \frac{Pc^2}{3EI}(b + c) \quad \text{(downward)}$$

(2) ASSUME THAT THE WIRE STRETCHES

T = tensile force in the wire

$$= \frac{P}{b}(b + c)$$

$$\delta_B = \frac{Th}{EA} = \frac{Ph(b + c)}{EAb}$$

$$\delta''_C = \delta_B\left(\frac{b + c}{b}\right) = \frac{Ph(b + c)^2}{EAb^2} \quad \text{(downward)}$$

(3) DEFLECTION AT POINT C

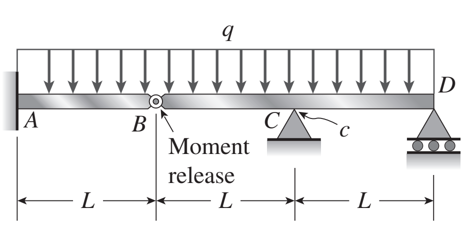
$$\delta_C = \delta'_c + \delta''_c = P(b + c)\left[\frac{c^2}{3EI} + \frac{h(b + c)}{EAb^2}\right] \quad \leftarrow$$

Substitute numerical values:

$$\delta_C = 0.10 \text{ in.} + 0.02 \text{ in.} = 0.12 \text{ in.} \quad \leftarrow$$

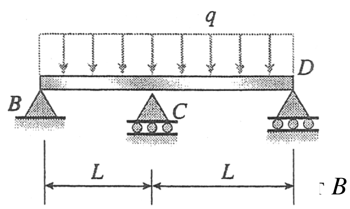
Problem 9.5-20 The compound beam shown in the figure consists of a cantilever beam AB (length L) that is pin-connected to a simple beam BD (length $2L$). After the beam is constructed, a clearance c exists between the beam and a support at C, midway between points B and D. Subsequently, a uniform load is placed along the entire length of the beam.

What intensity q of the load is needed to close the gap at C and bring the beam into contact with the support?



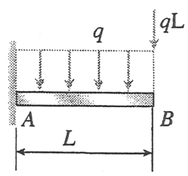
Solution 9.5-20 Compound beam

BEAM BCD WITH A SUPPORT AT B



$$\delta_c' = \frac{5q(2L)^4}{384EI} = \frac{5qL^4}{24EI}$$

CANTILEVER BEAM AB



$$\delta_B = \frac{qL^4}{8EI} + \frac{(qL)L^3}{3EI} = \frac{11qL^4}{24EI} \quad \text{(downward)}$$

δ_c'' = downward displacement of point C due to δ_B

$$\delta_c'' = \frac{1}{2}\delta_B = \frac{11qL^4}{48EI}$$

DOWNWARD DISPLACEMENT OF POINT C

$$\delta_c = \delta_c' + \delta_c'' = \frac{5qL^4}{24EI} + \frac{11qL^4}{48EI} = \frac{7qL^4}{16EI}$$

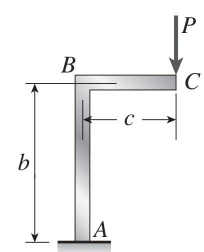
c = clearance $\qquad c = \delta_C = \dfrac{7qL^4}{16EI}$

INTENSITY OF LOAD TO CLOSE THE GAP

$$q = \frac{16EIc}{7L^4} \quad \leftarrow$$

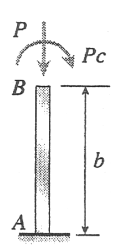
Problem 9.5-21 Find the horizontal deflection δ_h and vertical deflection δ_v at the free end C of the frame ABC shown in the figure. (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the load P.



Solution 9.5-21 Frame *ABC*

MEMBER AB:



δ_h = horizontal deflection of point B

Table G-1, Case 6:

$$\delta_h = \frac{(Pc)b^2}{2EI} = \frac{Pcb^2}{2EI}$$

$$\theta_B = \frac{Pcb}{EI}$$

Since member BC does not change in length,

δ_h is also the horizontal displacement of point C.

$$\therefore \delta_h = \frac{Pcb^2}{2EI} \quad \leftarrow$$

MEMBER BC WITH B FIXED AGAINST ROTATION:

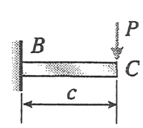


Table G-1, Case 4:

$$\delta_c' = \frac{Pc^3}{3EI}$$

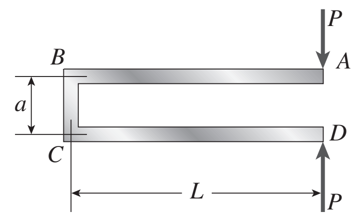
VERTICAL DEFLECTION OF POINT C

$$\delta_c = \delta_v = \delta_c' + \theta_B c = \frac{Pc^3}{3EI} + \frac{Pcb}{EI}(c) = \frac{Pc^2}{3EI}(c + 3b)$$

$$\delta_v = \frac{Pc^2}{3EI}(c + 3b) \quad \leftarrow$$

Problem 9.5-22 The frame *ABCD* shown in the figure is squeezed by two collinear forces *P* acting at points *A* and *D*. What is the decrease δ in the distance between points *A* and *D* when the loads *P* are applied? (The flexural rigidity *EI* is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the loads *P*.



Solution 9.5-22 Frame *ABCD*

MEMBER *BC*:

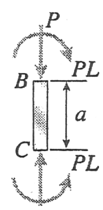


Table G-2, Case 10: $\theta_B = \dfrac{(PL)a}{2EI} = \dfrac{PLa}{2EI}$

MEMBER *BA*:

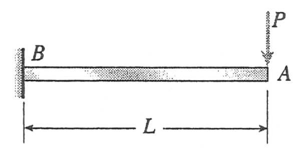


Table G-1, Case 4: $\delta_A = \dfrac{PL^3}{3EI} + \theta_B L$

$$= \frac{PL^3}{3EI} + \frac{PLa}{2EI}(L)$$

$$= \frac{PL^2}{6EI}(2L + 3a)$$

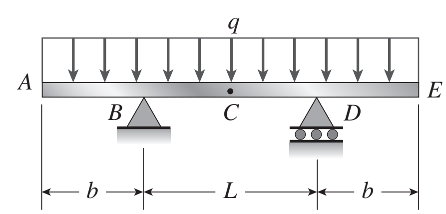
DECREASE IN DISTANCE BETWEEN POINTS *A* AND *D*

$$\delta = 2\delta_A = \frac{PL^2}{3EI}(2L + 3a) \quad \leftarrow$$

Problem 9.5-23 A beam *ABCDE* has simple supports at *B* and *D* and symmetrical overhangs at each end (see figure). The center span has length *L* and each overhang has length *b*. A uniform load of intensity *q* acts on the beam.

(a) Determine the ratio *b/L* so that the deflection δ_C at the midpoint of the beam is equal to the deflections δ_A and δ_E at the ends.

(b) For this value of *b/L*, what is the deflection δ_C at the midpoint?



Solution 9.5-23 Beam with overhangs

BEAM *BCD*:

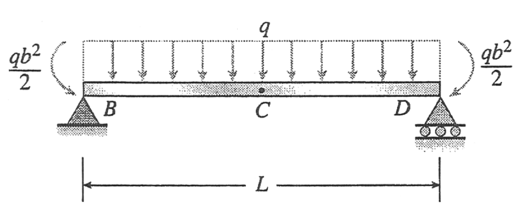


Table G-2, Case 1 and Case 10:

$$\theta_B = \frac{qL^3}{24EI} - \frac{qb^2}{2}\left(\frac{L}{2EI}\right)$$

$$= \frac{qL}{24EI}(L^2 - 6b^2) \quad \text{(clockwise is positive)}$$

$$\delta_C = \frac{5qL^4}{384EI} - \frac{qb^2}{2}\left(\frac{L^2}{8EI}\right) = \frac{qL^2}{384EI}(5L^2 - 24b^2)$$

(downward is positive) (1)

BEAM *AB*:

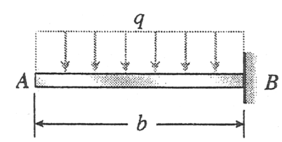


Table G-1, Case 1:

$$\delta_A = \frac{qb^4}{8EI} - \theta_B b = \frac{qb^4}{8EI} - \frac{qL}{24EI}(L^2 - 6b^2)b$$

$$= \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

(downward is positive)

DEFLECTION δ_C EQUALS DEFLECTION δ_A

$$\frac{qL^2}{384EI}(5L^2 - 24b^2) = \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

Rearrange and simplify the equation:

$$48b^4 + 96b^3L + 24b^2L^2 - 16bL^3 - 5L^4 = 0$$

or

$$48\left(\frac{b}{L}\right)^4 + 96\left(\frac{b}{L}\right)^3 + 24\left(\frac{b}{L}\right)^2 - 16\left(\frac{b}{L}\right) - 5 = 0$$

(a) RATIO $\dfrac{b}{L}$

Solve the preceding equation numerically:

$$\frac{b}{L} = 0.40301 \quad \text{Say,} \quad \frac{b}{L} = 0.4030 \quad \leftarrow$$

(b) DEFLECTION δ_C (EQ. 1)

$$\delta_C = \frac{qL^2}{384EI}(5L^2 - 24b^2)$$

$$= \frac{qL^2}{384EI}[5L^2 - 24\,(0.40301\,L)^2]$$

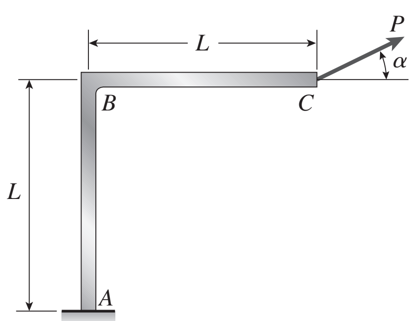
$$= 0.002870\frac{qL^4}{EI}$$

(downward deflection) ←

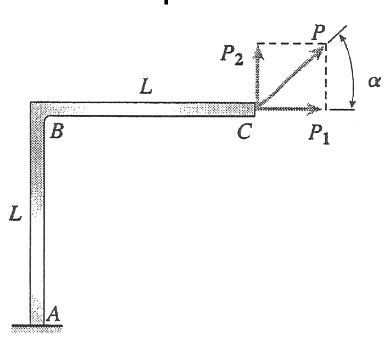
Problem 9.5-24 A frame *ABC* is loaded at point *C* by a force *P* acting at an angle α to the horizontal (see figure). Both members of the frame have the same length and the same flexural rigidity.

Determine the angle α so that the deflection of point *C* is in the same direction as the load. (Disregard the effects of axial deformations and consider only the effects of bending due to the load *P*.)

Note: A direction of loading such that the resulting deflection is in the same direction as the load is called a *principal direction*. For a given load on a planar structure, there are two principal directions, perpendicular to each other.



Solution 9.5-24 Principal directions for a frame



P_1 and P_2 are the components of the load P

$P_1 = P\cos\alpha$

$P_2 = P\sin\alpha$

IF P_1 ACTS ALONE $\quad \delta_H' = \dfrac{P_1L^3}{3EI}$ (to the right)

$$\delta_v' = \theta_B L = \left(\frac{P_1L^2}{2EI}\right)L = \frac{P_1L^3}{2EI}$$

(downward)

IF P_2 ACTS ALONE $\quad \delta_H'' = \dfrac{P_2L^3}{2EI}$ (to the left)

$$\delta_v'' = \frac{P_2L^3}{3EI} + \theta_B L = \frac{P_2L^3}{3EI} + \left(\frac{P_2L^2}{EI}\right)L = \frac{4P_2L^3}{3EI}$$

(upward)

DEFLECTIONS DUE TO THE LOAD P

$$\delta_H = \frac{P_1L^3}{3EI} - \frac{P_2L^3}{2EI} = \frac{L^3}{6EI}(2P_1 - 3P_2)$$

(to the right)

$$\delta_v = -\frac{P_1L^3}{2EI} + \frac{4P_2L^3}{3EI} = \frac{L^3}{6EI}(-3P_1 + 8P_2)$$

(upward)

$$\frac{\delta_v}{\delta_H} = \frac{-3P_1 + 8P_2}{2P_1 - 3P_2}$$

$$= \frac{-3P\cos\alpha + 8P\sin\alpha}{2P\cos\alpha - 3P\sin\alpha} = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

PRINCIPAL DIRECTIONS

The deflection of point C is in the same direction as the load P.

$$\therefore \tan\alpha = \frac{P_2}{P_1} = \frac{\delta_v}{\delta_H} \quad \text{or} \quad \tan\alpha = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

Rearrange and simplity: $\tan^2\alpha + 2\tan\alpha - 1 = 0$ (quadratic equation)

Solving, $\tan\alpha = -1 \pm \sqrt{2}$

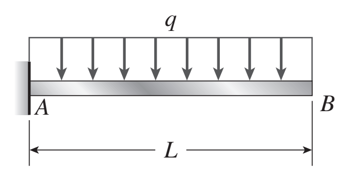
$\alpha = 22.5^\circ, \quad 112.5^\circ, \quad -67.5^\circ, \quad -157.5^\circ \quad \leftarrow$

Moment-Area Method

The problems for Section 9.6 are to be solved by the moment-area method. All beams have constant flexural rigidity EI.

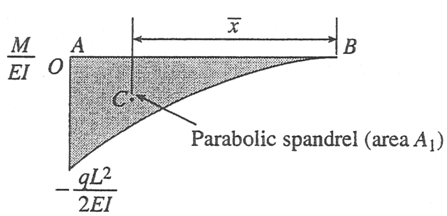
Problem 9.6-1 A cantilever beam AB is subjected to a uniform load of intensity q acting throughout its length (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-1 Cantilever beam (uniform load)

$\dfrac{M}{EI}$ DIAGRAM:



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 18: $A_1 = \dfrac{1}{3}(L)\left(\dfrac{qL^2}{2EI}\right) = \dfrac{qL^3}{6EI}$

$$\bar{x} = \frac{3L}{4}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{qL^3}{6EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{qL^3}{6EI} \quad \text{(clockwise)} \quad \leftarrow$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

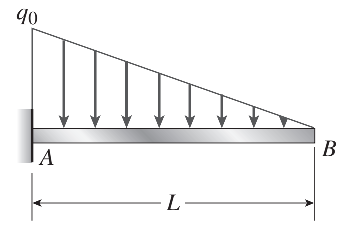
$$Q_1 = A_1\bar{x} = \left(\frac{qL^3}{6EI}\right)\left(\frac{3L}{4}\right) = \frac{qL^4}{8EI}$$

$$\delta_B = Q_1 = \frac{qL^4}{8EI} \quad \text{(Downward)} \quad \leftarrow$$

(These results agree with Case 1, Table G-1.)

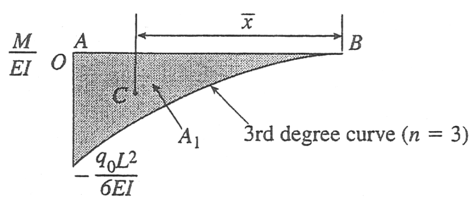
Problem 9.6-2 The load on a cantilever beam AB has a triangular distribution with maximum intensity q_0 (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-2 Cantilever beam (triangular load)

$\frac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 20:

$$A_1 = \frac{bh}{n+1} = \frac{1}{4}(L)\left(\frac{q_0L^2}{6EI}\right) = \frac{q_0L^3}{24EI}$$

$$\bar{x} = \frac{b(n+1)}{n+2} = \frac{4L}{5}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{q_0L^3}{24EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{q_0L^3}{24EI} \quad \text{(clockwise)} \quad \leftarrow$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

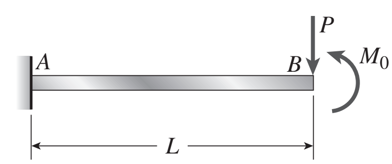
$$Q_1 = A_1\bar{x} = \left(\frac{q_0L^3}{24EI}\right)\left(\frac{4L}{5}\right) = \frac{q_0L^4}{30EI}$$

$$\delta_B = Q_1 = \frac{q_0L^4}{30EI} \quad \text{(Downward)} \quad \leftarrow$$

(These results agree with Case 8, Table G-1.)

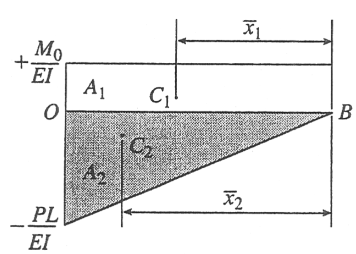
Problem 9.6-3 A cantilever beam AB is subjected to a concentrated load P and a couple M_0 acting at the free end (see figure).

Obtain formulas for the angle of rotation θ_B and the deflection δ_B at end B.



Solution 9.6-3 Cantilever beam (force *P* and couple M_0)

$\frac{M}{EI}$ DIAGRAM



NOTE: A_1 is the M/EI diagram for M_0 (rectangle). A_2 is the M/EI diagram for P (triangle).

ANGLE OF ROTATION

Use the sign conventions for the moment-area theorems (page 713 of textbook).

$$A_1 = \frac{M_0L}{EI} \qquad \bar{x}_1 = \frac{L}{2} \qquad A_2 = -\frac{PL^2}{2EI} \qquad \bar{x}_2 = \frac{2L}{3}$$

$$A_0 = A_1 + A_2 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0 \qquad \theta_A = 0$$

$$\theta_B = A_0 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

(θ_B is positive when counterclockwise)

DEFLECTION

Q = first moment of areas A_1 and A_2 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

$$t_{B/A} = Q = \delta_B \qquad \delta_B = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

(δ_B is positive when upward)

FINAL RESULTS

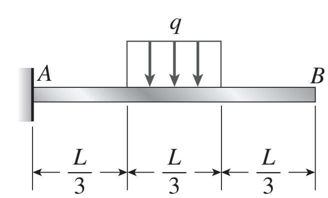
To match the sign conventions for θ_B and δ_B used in Appendix G, change the signs as follows.

$$\theta_B = \frac{PL^2}{2EI} - \frac{M_0 L}{EI} \quad \text{(positive clockwise)} \quad \leftarrow$$

$$\delta_B = \frac{PL^3}{3EI} - \frac{M_0 L^2}{2EI} \quad \text{(positive downward)} \quad \leftarrow$$

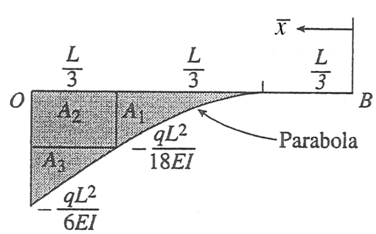
(These results agree with Cases 4 and 6, Table G-1.)

Problem 9.6-4 Determine the angle of rotation θ_B and the deflection δ_B at the free end of a cantilever beam AB with a uniform load of intensity q acting over the middle third of the length (see figure).



Solution 9.6-4 Cantilever beam with partial uniform load

$\frac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas. Appendix D, Cases 1, 6, and 18:

$$A_1 = \frac{1}{3}\left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{162EI}$$

$$\bar{x}_1 = \frac{L}{3} + \frac{3}{4}\left(\frac{L}{3}\right) = \frac{7L}{12}$$

$$A_2 = \left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{54EI} \qquad \bar{x}_2 = \frac{2L}{3} + \frac{L}{6} = \frac{5L}{6}$$

$$A_3 = \frac{1}{2}\left(\frac{L}{3}\right)\left(\frac{qL^2}{9EI}\right) = \frac{qL^3}{54EI}$$

$$\bar{x}_3 = \frac{2L}{3} + \frac{2}{3}\left(\frac{L}{3}\right) = \frac{8L}{9}$$

$$A_0 = A_1 + A_2 + A_3 = \frac{7qL^3}{162EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0$$

$$\theta_A = 0 \qquad \theta_B = \frac{7qL^3}{162EI} \quad \text{(clockwise)} \quad \leftarrow$$

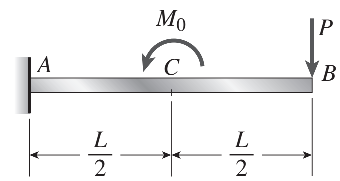
DEFLECTION

Q = first moment of area A_0 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = \frac{23qL^4}{648EI}$$

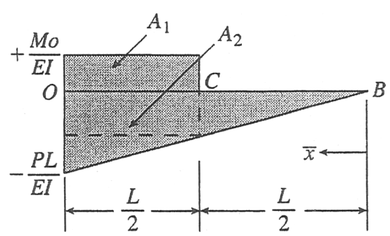
$$\delta_B = Q = \frac{23qL^4}{648EI} \quad \text{(Downward)} \quad \leftarrow$$

Problem 9.6-5 Calculate the deflections δ_B and δ_C at points B and C, respectively, of the cantilever beam ACB shown in the figure. Assume $M_0 = 36$ k-in., $P = 3.8$ k, $L = 8$ ft, and $EI = 2.25 \times 10^9$ lb-in.^{2}



Solution 9.6-5 Cantilever beam (force *P* and couple M_0)

$\frac{M}{EI}$ DIAGRAM



NOTE: A_1 is the *M/EI* diagram for M_0 (rectangle). A_2 is the *M/EI* diagram for P (triangle).

Use the sign conventions for the moment-area theorems (page 713 of textbook).

DEFLECTION δ_B

Q_B = first moment of areas A_1 and A_2 with respect to point B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) - \frac{1}{2}\left(\frac{PL}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{L^2}{24EI}(9M_0 - 8PL)$$

$$t_{B/A} = Q_B = \delta_B \qquad \delta_B = \frac{L^2}{24EI}(9M_0 - 8PL)$$

(δ_B is positive when upward)

DEFLECTION δ_C

Q_C = first moment of areas A_1 and left-hand part of A_2 with respect to point C

$$= \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{L^2}{48EI}(6M_0 - 5PL)$$

$$t_{C/A} = Q_C = \delta_C \qquad \delta_C = \frac{L^2}{48EI}(6M_0 - 5PL)$$

(δ_C is positive when upward)

ASSUME DOWNWARD DEFLECTIONS ARE POSITIVE (change the signs of δ_B and δ_C)

$$\delta_B = \frac{L^2}{24EI}(8PL - 9M_0) \qquad \leftarrow$$

$$\delta_C = \frac{L^2}{48EI}(5PL - 6M_0) \qquad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$M_0 = 36$ k-in. $\qquad P = 3.8$ k

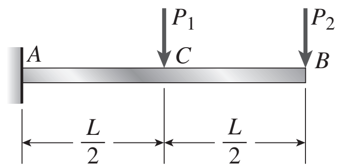
$L = 8$ ft $= 96$ in. $\qquad EI = 2.25 \times 10^6$ k-in.^{2}

$\delta_B = 0.4981$ in. $- 0.0553$ in. $= 0.443$ in. $\qquad \leftarrow$

$\delta_C = 0.1556$ in. $- 0.0184$ in. $= 0.137$ in. $\qquad \leftarrow$

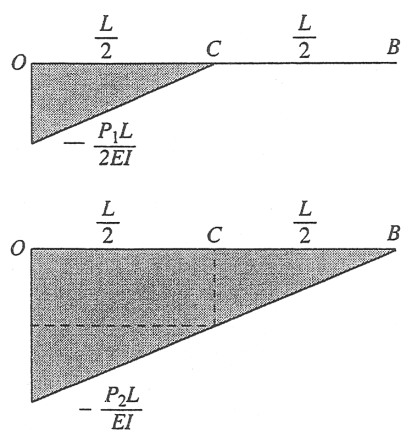
Problem 9.6-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2 as shown in the figure.

Calculate the deflections δ_B and δ_C at points B and C, respectively. Assume $P_1 = 10$ kN, $P_2 = 5$ kN, $L = 2.6$ m, $E = 200$ GPa, and $I = 20.1 \times 10^6$ mm^4.



Solution 9.6-6 Cantilever beam (forces P_1 and P_2)

$\frac{M}{EI}$ DIAGRAMS



$P_1 = 10$ kN $\quad P_2 = 5$ kN $\quad L = 2.6$ m

$E = 200$ GPa $\quad I = 20.1 \times 10^6$ mm^4

Use absolute values of areas.

DEFLECTION δ_B

$\delta_B = t_{B/A} = Q_B =$ first moment of areas with respect to point B

$$\delta_B = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2} + \frac{L}{3}\right) + \frac{1}{2}\left(\frac{P_2L}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{5P_1L^3}{48EI} + \frac{P_2L^3}{3EI} \quad \text{(downward)} \quad \leftarrow$$

DEFLECTION δ_C

$\delta_C = t_{C/A} = Q_C =$ first moment of areas to the left of point C with respect to point C

$$\delta_c = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) + \left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right)$$

$$+ \frac{1}{2}\left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{P_1L^3}{24EI} + \frac{5P_2L^3}{48EI} \quad \text{(downward)} \quad \leftarrow$$

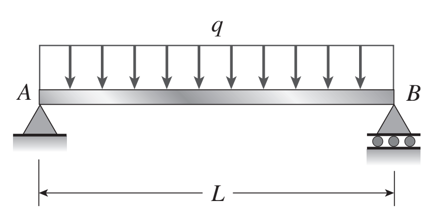
SUBSTITUTE NUMERICAL VALUES:

$\delta_B = 4.554$ mm $+ 7.287$ mm $= 11.84$ mm $\quad \leftarrow$

$\delta_C = 1.822$ mm $+ 2.277$ mm $= 4.10$ mm $\quad \leftarrow$

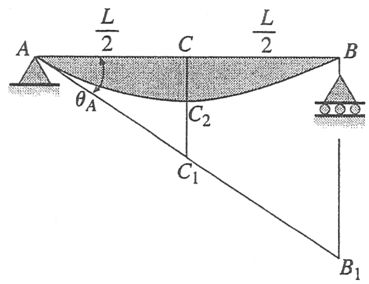
(deflections are downward)

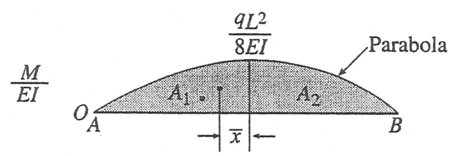
Problem 9.6-7 Obtain formulas for the angle of rotation θ_A at support A and the deflection δ_{max} at the midpoint for a simple beam AB with a uniform load of intensity q (see figure).



Solution 9.6-7 Simple beam with a uniform load

DEFLECTION CURVE AND $\frac{M}{EI}$ DIAGRAM





δ_{max} = maximum deflection (distance CC_2)

Use absolute values of areas.

ANGLE OF ROTATION AT END A

Appendix D, Case 17:

$$A_1 = A_2 = \frac{2}{3}\left(\frac{L}{2}\right)\left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{24EI}$$

$$\bar{x}_1 = \frac{3}{8}\left(\frac{L}{2}\right) = \frac{3L}{16}$$

$t_{B/A} = BB_1$ = first moment of areas A_1 and A_2 with respect to point B

$$= (A_1 + A_2)\left(\frac{L}{2}\right) = \frac{qL^4}{24EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{qL^3}{24EI} \quad \text{(clockwise)} \quad \leftarrow$$

DEFLECTION δ_{max} AT THE MIDPOINT C

$$\text{Distance } CC_1 = \frac{1}{2}(BB_1) = \frac{qL^4}{48EI}$$

$t_{C_2/A} = C_2C_1$ = first moment of area A_1 with respect to point C

$$= A_1\bar{x}_1 = \left(\frac{qL^3}{24EI}\right)\left(\frac{3L}{16}\right) = \frac{qL^4}{128EI}$$

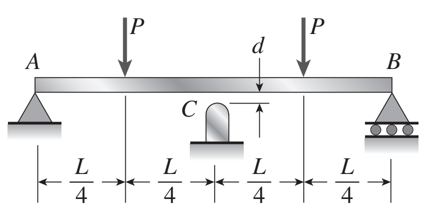
$$\delta_{max} = CC_2 = CC_1 - C_2C_1 = \frac{qL^4}{48EI} - \frac{qL^4}{128EI}$$

$$= \frac{5qL^4}{384EI} \quad \text{(downward)} \quad \leftarrow$$

(These results agree with Case 1 of Table G-2.)

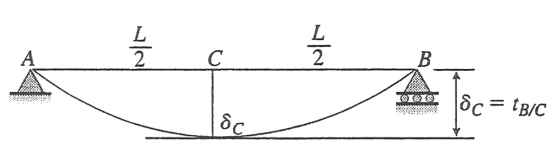
Problem 9.6-8 A simple beam AB supports two concentrated loads P at the positions shown in the figure. A support C at the midpoint of the beam is positioned at distance d below the beam before the loads are applied.

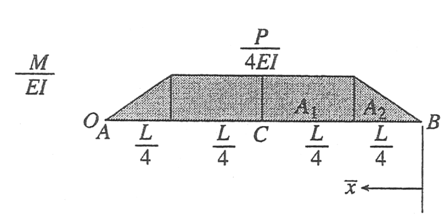
Assuming that $d = 10$ mm, $L = 6$ m, $E = 200$ GPa, and $I = 198 \times 10^6$ mm^4, calculate the magnitude of the loads P so that the beam just touches the support at C.



Solution 9.6-8 Simple beam with two equal loads

DEFLECTION CURVE AND $\frac{M}{EI}$ DIAGRAM





δ_C = deflection at the midpoint C

$$A_1 = \frac{PL^2}{16EI} \qquad \bar{x}_1 = \frac{3L}{8}$$

$$A_2 = \frac{PL^2}{32EI} \qquad \bar{x}_2 = \frac{L}{6}$$

Use absolute values of areas.

DEFLECTION δ_C AT MIDPOINT OF BEAM

At point C, the deflection curve is horizontal.

$\delta_C = t_{B/C} =$ first moment of area between B and C with respect to B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{PL^2}{16EI}\left(\frac{3L}{8}\right) + \frac{PL^2}{32EI}\left(\frac{L}{6}\right)$$

$$= \frac{11PL^3}{384EI}$$

$d =$ gap between the beam and the support at C

MAGNITUDE OF LOAD TO CLOSE THE GAP

$$\delta = d = \frac{11PL^3}{384EI} \qquad P = \frac{384EId}{11L^3} \qquad \leftarrow$$

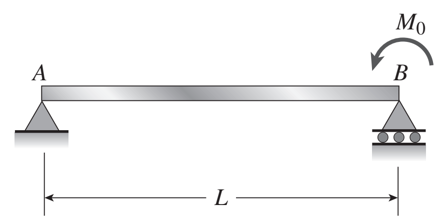
SUBSTITUTE NUMERICAL VALUES:

$d = 10$ mm $\qquad L = 6$ m $\qquad E = 200$ GPa

$I = 198 \times 10^6$ mm^4 $\qquad P = 64$ kN $\qquad \leftarrow$

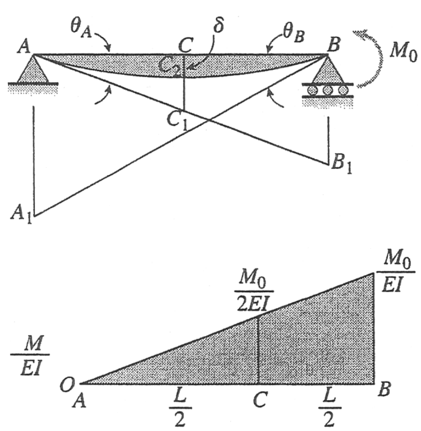
Problem 9.6-9 A simple beam AB is subjected to a load in the form of a couple M_0 acting at end B (see figure).

Determine the angles of rotation θ_A and θ_B at the supports and the deflection δ at the midpoint.



Solution 9.6-9 Simple beam with a couple M_0

DEFLECTION CURVE AND $\dfrac{M}{EI}$ DIAGRAM



$\delta =$ deflection at the midpoint C

$\delta =$ distance CC_2

Use absolute values of areas.

ANGLE OF ROTATION θ_A

$t_{B/A} = BB_1 =$ first moment of area between A and B with respect to B

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{L}{3}\right) = \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6EI} \quad \text{(clockwise)} \qquad \leftarrow$$

ANGLE OF ROTATION θ_B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{2L}{3}\right) = \frac{M_0L^2}{3EI}$$

$$\theta_B = \frac{AA_1}{L} = \frac{M_0L}{3EI} \quad \text{(Counterclockwise)} \qquad \leftarrow$$

DEFLECTION δ AT THE MIDPOINT C

Distance $CC_1 = \dfrac{1}{2}(BB_1) = \dfrac{M_0L^2}{12EI}$

$t_{C_2/A} = C_2C_1 =$ first moment of area between A and C with respect to C

$$= \frac{1}{2}\left(\frac{M_0}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = \frac{M_0L^2}{48EI}$$

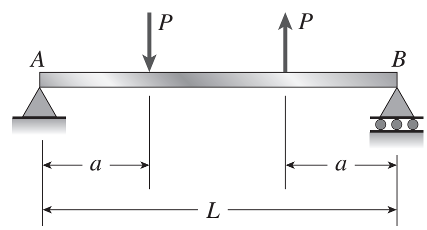
$$\delta = CC_1 - C_2C_1 = \frac{M_0L^2}{12EI} - \frac{M_0L^2}{48EI}$$

$$= \frac{M_0L^2}{16EI} \quad \text{(Downward)} \quad \leftarrow$$

(These results agree with Case 7 of Table G-2.)

Problem 9.6-10 The simple beam AB shown in the figure supports two equal concentrated loads P, one acting downward and the other upward.

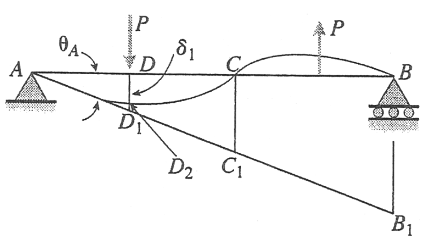
Determine the angle of rotation θ_A at the left-hand end, the deflection δ_1 under the downward load, and the deflection δ_2 at the midpoint of the beam.

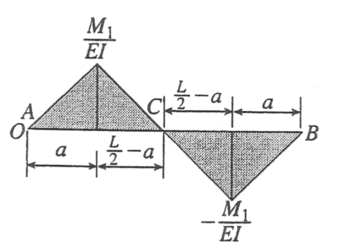


Solution 9.6-10 Simple beam with two loads

Because the beam is symmetric and the load is antisymmetric, the deflection at the midpoint is zero.

$\therefore \delta_2 = 0 \quad \leftarrow$





$$\frac{M_1}{EI} = \frac{Pa(L-2a)}{LEI}$$

$$A_1 = \frac{1}{2}\left(\frac{M_1}{EI}\right)(a) = \frac{Pa^2(L-2a)}{2LEI}$$

$$A_2 = \frac{1}{2}\left(\frac{M_1}{EI}\right)\left(\frac{L}{2} - a\right) = \frac{Pa(L-2a)^2}{4LEI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{C/A} = CC_1 =$ first moment of area between A and C with respect to C

$$= A_1\left(\frac{L}{2} - a + \frac{a}{3}\right) + A_2\left(\frac{2}{3}\right)\left(\frac{L}{2} - a\right)$$

$$= \frac{Pa(L-a)(L-2a)}{12EI}$$

$$\theta_A = \frac{CC_1}{L/2} = \frac{Pa(L-a)(L-2a)}{6LEI} \quad \text{(clockwise)} \quad \leftarrow$$

DEFLECTION δ_1 UNDER THE DOWNWARD LOAD

Distance $DD_1 = \left(\dfrac{a}{L/2}\right)(CC_1)$

$$= \frac{Pa^2(L-a)(L-2a)}{6LEI}$$

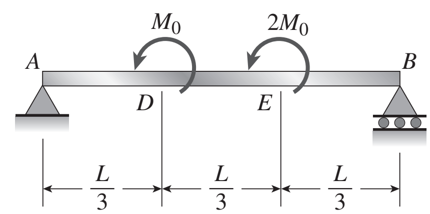
$t_{D_2/A} = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{a}{3}\right) = \frac{Pa^3(L-2a)}{6LEI}$$

$$\delta_1 = DD_1 - D_2D_1$$

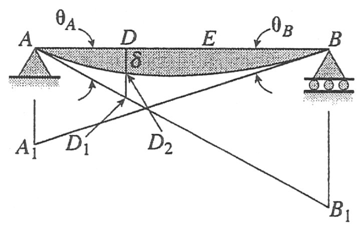
$$= \frac{Pa^2(L-2a)^2}{6LEI} \quad \text{(Downward)} \quad \leftarrow$$

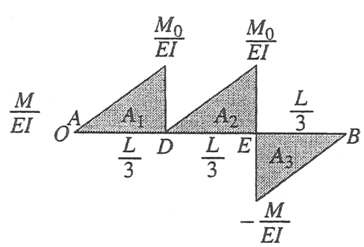
Problem 9.6-11 A simple beam AB is subjected to couples M_0 and $2M_0$ as shown in the figure. Determine the angles of rotation θ_A and θ_B at the beam and the deflection δ at point D where the load M_0 is applied.



Solution 9.6-11 Simple beam with two couples

DEFLECTION CURVE AND $\dfrac{M}{EI}$ DIAGRAM





$$A_1 = A_2 = \frac{1}{2}\left(\frac{M_0}{EI}\right)\left(\frac{L}{3}\right) = \frac{M_0L}{6EI} \qquad A_3 = -\frac{M_0L}{6EI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{B/A} = BB_1 =$ first moment of area between A and B with respect to B

$$= A_1\left(\frac{2L}{3} + \frac{L}{9}\right) + A_2\left(\frac{L}{3} + \frac{L}{9}\right) + A_3\left(\frac{2L}{9}\right)$$

$$= \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6EI} \quad \text{(clockwise)} \qquad \leftarrow$$

ANGLE OF ROTATION θ_B AT END B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= A_1\left(\frac{2L}{9}\right) + A_2\left(\frac{L}{3} + \frac{2L}{9}\right) + A_3\left(\frac{2L}{3} + \frac{L}{9}\right) = 0$$

$$\theta_B = \frac{AA_1}{L} = 0 \qquad \leftarrow$$

DEFLECTION δ AT POINT D

Distance $DD_1 = \dfrac{1}{3}(BB_1) = \dfrac{M_0L^2}{18EI}$

$t_{D_2/A} = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{L}{9}\right) = \frac{M_0L^2}{54EI}$$

$$\delta = DD_1 - D_2D_1 = \frac{M_0L^2}{27EI}$$

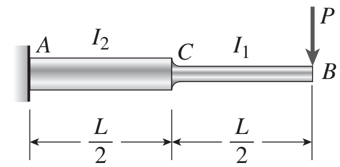
(downward) $\leftarrow$

NOTE: This deflection is also the maximum deflection.

Nonprismatic Beams

Problem 9.7-1 The cantilever beam ACB shown in the figure has moments of inertia I_2 and I_1 in parts AC and CB, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the load P.
(b) Determine the ratio r of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.
(c) Plot a graph of the deflection ratio r versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

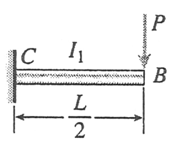


Solution 9.7-1 Cantilever beam (nonprismatic)

Use the method of superposition.

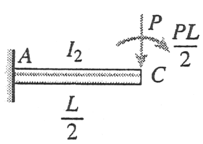
(a) Deflection δ_B at the free end

(1) Part CB of the beam:



$$(\delta_B)_1 = \frac{P}{3EI_1}\left(\frac{L}{2}\right)^3 = \frac{PL^3}{24EI_1}$$

(2) Part AC of the beam:



$$\delta_C = \frac{P(L/2)^3}{3EI_2} + \frac{(PL/2)(L/2)^2}{2EI_2} = \frac{5PL^3}{48EI_2}$$

$$\theta_C = \frac{P(L/2)^2}{2EI_2} + \frac{(PL/2)(L/2)}{EI_2} = \frac{3PL^2}{8EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{7PL^3}{24EI_2}$$

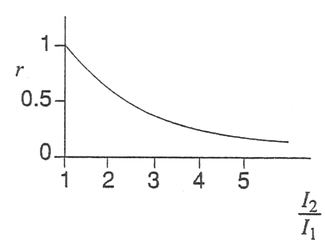
(3) Total deflection at point B

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{PL^3}{24EI_1}\left(1 + \frac{7I_1}{I_2}\right) \quad \leftarrow$$

(b) Prismatic beam $\delta_1 = \dfrac{PL^3}{3EI_1}$

Ratio: $r = \dfrac{\delta_B}{\delta_1} = \dfrac{1}{8}\left(1 + \dfrac{7I_1}{I_2}\right) \quad \leftarrow$

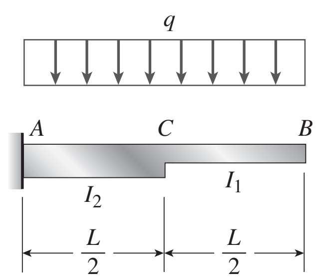
(c) Graph of ratio



$\frac{I_2}{I_1}$	r
1	1.00
2	0.56
3	0.42
4	0.34
5	0.30

Problem 9.7-2 The cantilever beam *ACB* shown in the figure supports a uniform load of intensity q throughout its length. The beam has moments of inertia I_2 and I_1 in parts *AC* and *CB*, respectively.

(a) Using the method of superposition, determine the deflection δ_B at the free end due to the uniform load.
(b) Determine the ratio r of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.
(c) Plot a graph of the deflection ratio r versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

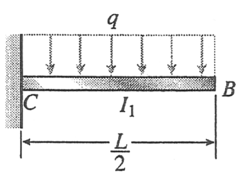


Solution 9.7-2 Cantilever beam (nonprismatic)

Use the method of superposition

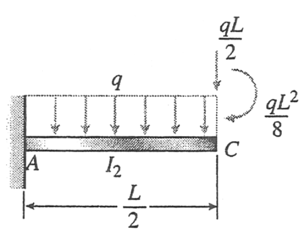
(a) DEFLECTION δ_B AT THE FREE END

(1) Part *CB* of the beam:



$$(\delta_B)_1 = \frac{q}{8EI_1}\left(\frac{L}{2}\right)^4 = \frac{qL^4}{128EI_1}$$

(2) Part *AC* of the beam:



$$\delta_C = \frac{q(L/2)^4}{8EI_2} + \frac{\left(\frac{qL}{2}\right)(L/2)^3}{3EI_2} + \frac{\left(\frac{qL^2}{8}\right)\left(\frac{L}{2}\right)^2}{2EI_2} = \frac{17qL^4}{384EI_2}$$

$$\theta_C = \frac{q(L/2)^3}{6EI_2} + \frac{(qL/2)(L/2)^2}{2EI_2} + \frac{(qL^2/8)(L/2)}{EI_2} = \frac{7qL^3}{48EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C\left(\frac{L}{2}\right) = \frac{15qL^4}{128EI_2}$$

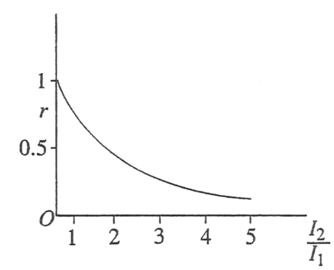
(3) Total deflection at point *B*

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qL^4}{128EI_1}\left(1 + \frac{15I_1}{I_2}\right) \quad \leftarrow$$

(b) PRISMATIC BEAM $\delta_1 = \dfrac{qL^4}{8EI_1}$

$$\text{Ratio: } r = \frac{\delta_B}{\delta_1} = \frac{1}{16}\left(1 + \frac{15I_1}{I_2}\right) \quad \leftarrow$$

(c) GRAPH OF RATIO

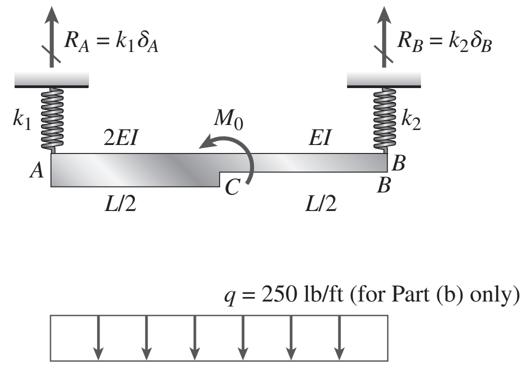


$\frac{I_2}{I_1}$	r
1	1.00
2	0.53
3	0.38
4	0.30
5	0.25

***Problem 9.7-3** Beam *ACB* hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity *EI*.

(a) What is the downward displacement of point *C*, which is at the midpoint of the beam, when the moment M_0 is applied? Data for the structure are as follows: $M_0 = 7.5$ k-ft, $L = 6$ ft, $EI = 520$ k-ft^2, $k_1 = 17$ k/ft, and $k_2 = 11$ k/ft.

(b) Repeat (a) but remove M_0 and, instead, apply uniform load q over the entire beam.



***Solution 9.7-3**

$M_0 = 7.5$ kip/ft $L = 6$ ft $EI = 520$ kip/ft^2 $k_1 = 17$ kip/ft $k_2 = 11$ kip/ft $q = 250$ lb/ft

(a) BENDING-MOMENT EQUATIONS-MOMENT M_0 AT C

$$2EIv'' = M = \frac{M_0 x}{L} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EIv' = \frac{M_0 x^2}{2L} + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EIv = \frac{M_0 x^3}{6L} + C_1 x + C_2 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. $v(0) = 0$ $C_2 = 0$ $2EIv = \dfrac{M_0 x^3}{6L} + C_1 x \quad \left(0 \le x \le \dfrac{L}{2}\right)$

$$EIv'' = -\frac{M_0}{2} + \frac{M_0\left(x - \frac{L}{2}\right)}{L} = -M_0 + \frac{M_0 x}{L} \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = -M_0 x + \frac{M_0 x^2}{2L} + C_3 \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_3 x + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. $v(L) = 0$ $-\dfrac{M_0 L^2}{2} + \dfrac{M_0 L^3}{6L} + C_3 L + C_4 = 0$ (1)

B.C. $v'_L\left(\dfrac{L}{2}\right) = v'_R\left(\dfrac{L}{2}\right)$ $\dfrac{1}{2}\left[\dfrac{M_0\left(\frac{L}{2}\right)^2}{2L} + C_1\right] = -M_0\dfrac{L}{2} + \dfrac{M_0\left(\frac{L}{2}\right)^2}{2L} + C_3$ (2)

B.C. $v_L\left(\dfrac{L}{2}\right) = v_R\left(\dfrac{L}{2}\right)$ $\dfrac{1}{2}\left[\dfrac{M_0\left(\frac{L}{2}\right)^3}{6L} + C_1\dfrac{L}{2}\right] = -\dfrac{M_0\left(\frac{L}{2}\right)^2}{2} + \dfrac{M_0\left(\frac{L}{2}\right)^3}{6L} + C_3\dfrac{L}{2} + C_4$ (3)

From (1), (2), and (3)

$$C_1 = 0 \qquad C_3 = \frac{7}{16}M_0L \qquad C_4 = \frac{-5}{48}M_0L^2 \qquad \leftarrow$$

Therefore

$$v(x) = \frac{M_0x^3}{12EIL} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$v(x) = \frac{M_0}{48EIL}(-24x^2L + 8x^3 + 21L^2x - 5L^3) \quad \left(\frac{L}{2} \le x \le L\right)$$

DEFLECTION AT A AND B

$$R_A = \frac{M_0}{L} \qquad R_B = -\frac{M_0}{L}$$

$$\delta_A = \frac{R_A}{k_1} \qquad \delta_B = \frac{R_B}{k_2}$$

$\delta_A = 0.88$ in. Downward $\qquad \delta_B = -1.36$ in. Upward

DEFLECTION AT POINT C

$$\delta_C = -v\left(\frac{L}{2}\right) + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = -\frac{M_0\left(\frac{L}{2}\right)^3}{12EIL} + \frac{1}{2}(\delta_A + \delta_B)$$

$\delta_C = -0.31$ in. Upward $\qquad \leftarrow$

(b) BENDING-MOMENT EQUATIONS-UNIFORM LOAD q

$$2EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$2EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1x + C_2 \quad \left(0 \le x \le \frac{L}{2}\right)$$

B.C. $v(0) = 0 \qquad C_2 = 0 \qquad 2EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1x \quad \left(0 \le x \le \frac{L}{2}\right)$

$$EIv'' = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_3 \quad \left(\frac{L}{2} \le x \le L\right)$$

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_3x + C_4 \quad \left(\frac{L}{2} \le x \le L\right)$$

B.C. $v(L) = 0 \qquad \frac{qLL^3}{12} - \frac{qL^4}{24} + C_3L + C_4 = 0 \qquad (1)$

B.C. $v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right) \quad \frac{1}{2}\left[\frac{qL\left(\frac{L}{2}\right)^2}{4} - \frac{q\left(\frac{L}{2}\right)^3}{6} + C_1\right] = \frac{qL\left(\frac{L}{2}\right)^2}{4} - \frac{q\left(\frac{L}{2}\right)^3}{6} + C_3 \qquad (2)$

B.C. $v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right) \quad \frac{1}{2}\left[\frac{qL\left(\frac{L}{2}\right)^3}{12} - \frac{q\left(\frac{L}{2}\right)^4}{24} + C_1\frac{L}{2}\right]$

$$= \frac{qL\left(\frac{L}{2}\right)^3}{12} - \frac{q\left(\frac{L}{2}\right)^4}{24} + C_3\frac{L}{2} + C_4 \qquad (3)$$

From (1), (2), and (3)

$$C_1 = \frac{-7}{128}qL^3 \qquad C_3 = \frac{-37}{768}qL^3 \qquad C_4 = \frac{5}{768}qL^4$$

Therefore

$$v(x) = -\frac{qx}{768EI}(-32Lx^2 + 16x^3 + 21L^3) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$v(x) = \frac{q}{768EI}(64Lx^3 - 32x^4 - 37L^3x + 5L^4) \quad \left(\frac{L}{2} \le x \le L\right)$$

Deflection at A and B

$$R_A = \frac{qL}{2} \qquad R_B = \frac{qL}{2}$$

$$\delta_A = \frac{R_A}{k_1} \qquad \delta_B = \frac{R_B}{k_2}$$

$\delta_A = 0.53$ in. Downward $\qquad \delta_B = 0.82$ in. Downward

Deflection at point C

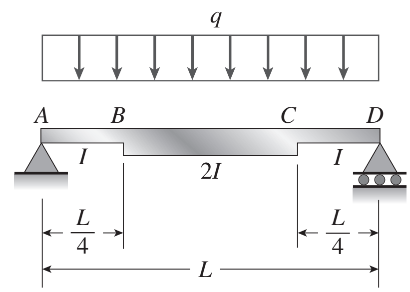
$$\delta_C = -v\left(\frac{L}{2}\right) + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = \frac{q\frac{L}{2}}{768EI}\left[-32L\left(\frac{L}{2}\right)^2 + 16\left(\frac{L}{2}\right)^3 + 21L^3\right] + \frac{1}{2}(\delta_A + \delta_B)$$

$\delta_C = 0.75$ in. Downward $\leftarrow$

Problem 9.7-4 A simple beam *ABCD* has moment of inertia I near the supports and moment of inertia $2I$ in the middle region, as shown in the figure. A uniform load of intensity q acts over the entire length of the beam.

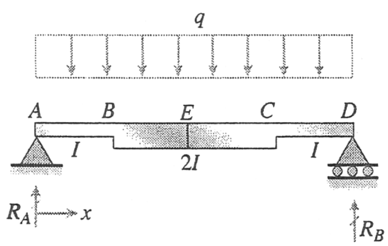
Determine the equations of the deflection curve for the left-hand half of the beam. Also, find the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.



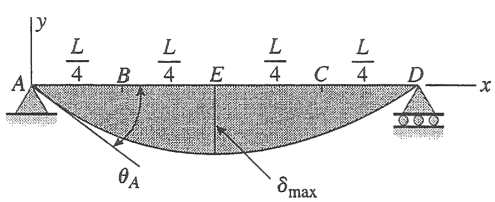
Solution 9.7-4 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-12a).

REACTIONS, BENDING MOMENT, AND DEFLECTION CURVE



$$R_A = R_B = \frac{qL}{2} \qquad M = Rx - \frac{qx^2}{2} = \frac{qLx}{2} - \frac{qx^2}{2}$$



BENDING-MOMENT EQUATIONS FOR THE LEFT-HAND HALF OF THE BEAM

$$EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{4}\right) \tag{1}$$

$$E(2I)v'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \le x \le \frac{L}{4}\right) \tag{3}$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_2 \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{4}$$

B.C. 1 Symmetry: $v'\left(\frac{L}{2}\right) = 0$

From Eq. (4): $C_2 = -\frac{qL^3}{24}$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \tag{5}$$

SLOPE AT POINT B (FROM THE RIGHT)

Substitute $x = \frac{L}{4}$ into Eq. (5):

$$EIv'_B = -\frac{11qL^3}{768} \tag{6}$$

B.C. 2 CONTINUITY OF SLOPES AT POINT B

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (6):

$$\frac{qL}{4}\left(\frac{L}{4}\right)^2 - \frac{q}{6}\left(\frac{L}{4}\right)^3 + C_1 = -\frac{11qL^3}{768} \qquad \therefore C_1 = -\frac{7qL^3}{256}$$

SLOPE OF THE BEAM (FROM EQS. 3 AND 5)

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{7qL^3}{256} \quad \left(0 \le x \le \frac{L}{4}\right) \tag{7}$$

$$EIv' = \frac{qLx^2}{8} - \frac{qx^3}{12} - \frac{qL^3}{48} \quad \left(\frac{L}{4} \le x \le \frac{L}{3}\right) \tag{8}$$

ANGLE OF ROTATION θ_A (FROM EQ. 7)

$$\theta_A = -v'(0) = \frac{7qL^3}{256EI} \text{ (positive clockwise)} \quad \leftarrow$$

INTEGRATE EQS. (7) AND (8)

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{7qL^3x}{256} + C_3 \quad \left(0 \le x \le \frac{L}{4}\right) \quad (9)$$

$$EIv = \frac{qLx^3}{24} - \frac{qx^4}{48} - \frac{qL^3x}{48} + C_4 \quad \left(\frac{L}{4} \le x \le \frac{L}{2}\right) \quad (10)$$

B.C. 3 Deflection at support A

$v(0) = 0$ From Eq. (9): $C_3 = 0$

DEFECTION AT POINT B (FROM THE LEFT)

Substitute $x = \frac{L}{4}$ into Eq. (9) with $C_3 = 0$

$$EIv_B = -\frac{35\,qL^4}{6144} \quad (11)$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Right}} = (v_B)_{\text{Left}}$

From Eqs. (10) and (11):

$$\frac{qL}{24}\left(\frac{L}{4}\right)^3 - \frac{q}{48}\left(\frac{L}{4}\right)^4 - \frac{qL^3}{48}\left(\frac{L}{4}\right) + C_4 = -\frac{35qL^4}{6144}$$

$$\therefore C_4 = -\frac{13qL^4}{12{,}288}$$

DEFECTION OF THE BEAM (FROM EQS. 9 AND 10)

$$v = -\frac{qx}{768EI}(21L^3 - 64Lx^2 + 32x^3)$$

$$\left(0 \le x \le \frac{L}{4}\right) \quad \leftarrow$$

$$v = -\frac{q}{12{,}288EI}(13L^4 + 256L^3x - 512Lx^3 + 256x^4)$$

$$\left(\frac{L}{4} \le x \le \frac{L}{2}\right) \quad \leftarrow$$

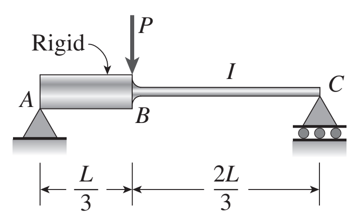
MAXIMUM DEFLECTION (AT THE MIDPOINT E)

(From the preceding equation for v.)

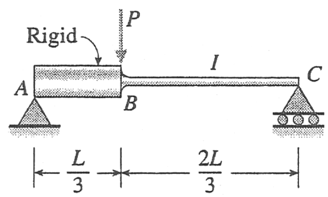
$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{31qL^4}{4096EI} \quad \text{(positive downward)} \quad \leftarrow$$

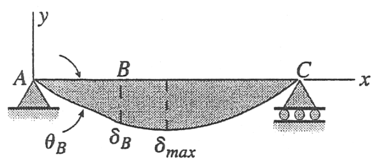
Problem 9.7-5 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection $\delta_{\max}$.



Solution 9.7-5 Simple beam with a rigid segment





FROM A TO B

$$v = -\frac{3\delta_B x}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$v' = -\frac{3\delta_B}{L} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{2}$$

FROM B TO C

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

B.C. 1 At $x = L/3$, $\quad v' = -\dfrac{3\delta_B}{L}$

$$\therefore C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \quad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \quad \left(\frac{L}{3} \le x \le L\right)$$

B.C. 2 $v(L) = 0 \qquad \therefore C_2 = -\dfrac{PL^3}{54} + 3EI\delta_B$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^2}{54} + 3EI\delta_B \quad \left(\frac{L}{3} \le x \le L\right) \tag{5}$$

B.C. 3 At $x = \dfrac{L}{3}$, $(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$ (Eqs. 1 and 5)

$$\therefore \delta_B = \frac{8PL^3}{729EI} \quad \longleftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \longleftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI}(7L^3 - 61L^2x + 81Lx^2 - 27x^3) \quad \left(\frac{L}{3} \le x \le L\right) \tag{6}$$

Also,

$$v' = \frac{P}{486EI}(-61L^2 + 162Lx - 81x^2) \quad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

MAXIMUM DEFLECTION

$v' = 0$ gives $x_1 = \dfrac{L}{9}(9 - 2\sqrt{5}) = 0.5031L$

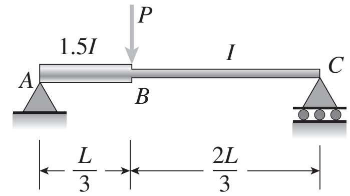
Substitute x_1 in Eq. (6) and simplify:

$$v_{\max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{\max} = -v_{\max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363\frac{PL^3}{EI} \quad \longleftarrow$$

Problem 9.7-6 A simple beam ABC has moment of inertia $1.5I$ from A to B and I from B to C (see figure). A concentrated load P acts at point B.

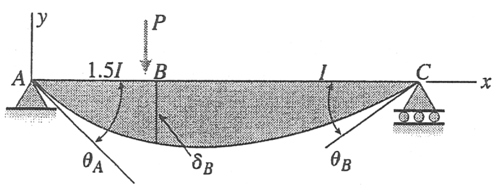
Obtain the equations of the deflection curves for both parts of the beam. From the equations, determine the angles of rotation θ_A and θ_C at the supports and the deflection δ_B at point B.



Solution 9.7-6 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-12a).

DEFLECTION CURVE



BENDING-MOMENT EQUATIONS

$$E\left(\frac{3I}{2}\right)v'' = M = \frac{2Px}{3} \quad \left(0 \le x \le \frac{L}{3}\right) \tag{1}$$

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \quad \left(\frac{L}{3} \le x \le L\right) \tag{2}$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{4Px^2}{18} + C_1 \quad \left(0 \le x \le \frac{L}{3}\right) \tag{3}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{2} + C_2 \quad \left(\frac{L}{3} \le x \le L\right) \tag{4}$$

B.C. 1 Continuity of slopes at point B

$(v'_B)_{\text{Left}} = (v'_B)_{\text{Right}}$

From Eqs. (3) and (4):

$$\frac{4P}{18}\left(\frac{L}{3}\right)^2 + C_1 = \frac{PL}{3}\left(\frac{L}{3}\right) - \frac{P}{6}\left(\frac{L}{3}\right)^2 + C_2$$

$$C_2 = C_1 - \frac{11PL^2}{162} \tag{5}$$

INTEGRATE EQS. (3) AND (4)

$$EIv = \frac{4Px^3}{54} + C_1x + C_3 \quad \left(0 \le x \le \frac{L}{3}\right) \tag{6}$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} + C_2x + C_4 \quad \left(\frac{L}{3} \le x \le L\right) \tag{7}$$

B.C. 2 Deflection at support A

$$v(0) = 0 \qquad \text{From Eq. (6):} \qquad C_3 = 0 \tag{8}$$

B.C. 3 Deflection at support C

$$v(L) = 0 \quad \text{From Eq. (7):} \qquad C_4 = -\frac{PL^3}{9} - C_2L \tag{9}$$

B.C. 4 Continuity of deflections at point B

$(v_B)_{\text{Left}} = (v_B)_{\text{Right}}$

From Eqs. (6), (8), and (7):

$$\frac{4P}{54}\left(\frac{L}{3}\right)^3 + C_1\left(\frac{L}{3}\right) = \frac{PL}{6}\left(\frac{L}{3}\right)^2 - \frac{P}{18}\left(\frac{L}{3}\right)^3 + C_2\left(\frac{L}{3}\right) + C_4$$

$$C_1L = \frac{10PL^3}{243} + C_2L + 3C_4 \tag{10}$$

SOLVE EQS. (5), (8), (9), AND (10)

$$C_1 = -\frac{38PL^2}{729} \qquad C_2 = -\frac{175PL^2}{1458} \qquad C_3 = 0$$

$$C_4 = \frac{13PL^3}{1458}$$

SLOPES OF THE BEAM (FROM EQS. 3 AND 4)

$$v' = -\frac{2P}{729EI}(19L^2 - 81x^2) \quad \left(0 \le x \le \frac{L}{3}\right) \tag{11}$$

$$v' = -\frac{P}{1458EI}(175L^2 - 486Lx + 243x^2)$$

$$\left(\frac{L}{3} \le x \le L\right) \qquad (12)$$

ANGLE OF ROTATION θ_A (FROM EQ. 11)

$$\theta_A = -v'(0) = \frac{38PL^2}{729EI} \quad \text{(positive clockwise)} \quad \longleftarrow$$

ANGLE OF ROTATION θ_C (FROM EQ. 12)

$$\theta_C = v'(L) = \frac{34PL^2}{729EI} \quad \text{(positive counterclockwise)} \quad \longleftarrow$$

DEFLECTIONS OF THE BEAM

Substitute C_1, C_2, C_3, and C_4 into Eqs. (6) and (7):

$$v = -\frac{2Px}{729EI}(19L^2 - 27x^2) \quad \left(0 \le x \le \frac{L}{3}\right) \quad \longleftarrow$$

$$v = -\frac{P}{1458EI}(-13L^3 + 175L^2x - 243Lx^2 + 81x^3)$$

$$\left(\frac{L}{3} \le x \le L\right) \quad \longleftarrow$$

DEFLECTION AT POINT B $\left(x = \frac{L}{3}\right)$

$$\delta_B = -v\left(\frac{L}{3}\right) = \frac{32PL^3}{2187EI} \quad \text{(positive downward)} \quad \longleftarrow$$

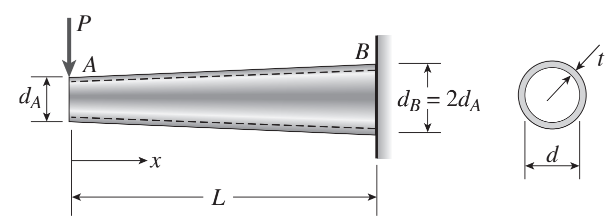
Problem 9.7-7 The tapered cantilever beam AB shown in the figure has thin-walled, hollow circular cross sections of constant thickness t. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi t d^3}{8} = \frac{\pi t d_A^3}{8L^3}(L + x)^3 = \frac{I_A}{L^3}(L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-7 Tapered cantilever beam

$$M = -Px \qquad EIv'' = -Px \qquad I = \frac{I_A}{L^3}(L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^3}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\displaystyle\int \frac{x\,dx}{(L + x)^3} = -\frac{L + 2x}{2(L + x)^2}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2(L + x)^2}\right] + C_1$$

B.C. 1 $v'(L) = 0 \qquad \therefore C_1 = -\dfrac{3PL^2}{8EI_A}$

$$v' = \frac{PL^3}{EI_A}\left[\frac{L + 2x}{2(L + x)^2}\right] - \frac{3PL^2}{8EI_A}$$

or

$$v' = \frac{PL^3}{EI_A}\left[\frac{L}{2(L + x)^2}\right] + \frac{PL^3}{EI_A}\left[\frac{x}{(L + x)^2}\right] - \frac{3PL^2}{8EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C:

$$\int \frac{dx}{(L + x)^2} = -\frac{1}{L + x}$$

$$\int \frac{x\,dx}{(L + x)^2} = \frac{L}{L + x} + \ln(L + x)$$

$$v = \frac{PL^3}{EI_A}\left(\frac{L}{2}\right)\left(-\frac{1}{L+x}\right) + \frac{PL^3}{EI_A}\left[\frac{L}{L+x} + \ln(L+x)\right] - \frac{3PL^2}{8EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{L}{2(L+x)} + \ln(L+x) - \frac{3x}{8L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0 \qquad \therefore C_2 = \frac{PL^3}{EI_A}\left[\frac{1}{8} - \ln(2L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{EI_A}\left[\frac{L}{2(L+x)} - \frac{3x}{8L} + \frac{1}{8} + \ln\left(\frac{L+x}{2L}\right)\right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{8EI_A}(8\ln 2 - 5)$$

$$= 0.06815\frac{PL^3}{EI_A} \text{ (positive downward)} \quad \leftarrow$$

Note: $\ln\frac{1}{2} = -\ln 2$

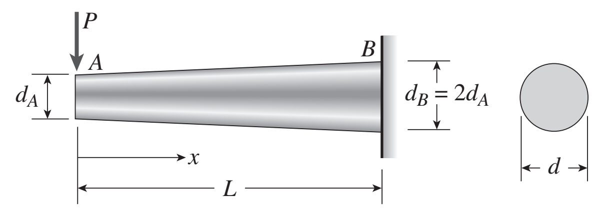
Problem 9.7-8 The tapered cantilever beam AB shown in the figure has a solid circular cross section. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L+x)$$

$$I = \frac{\pi d^4}{64} = \frac{\pi d_A^4}{64L^4}(L+x)^4 = \frac{I_A}{L^4}(L+x)^4$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-8 Tapered cantilever beam

$M = -Px \qquad EIv'' = -Px \qquad I = \frac{I_A}{L^4}(L+x)^4$

$$v'' = -\frac{Px}{EI} = -\frac{PL^4}{EI_A}\left[\frac{x}{(L+x)^4}\right] \qquad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $\int \frac{x\,dx}{(L+x)^4} = -\frac{L+3x}{6(L+x)^3}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L+3x}{6(L+x)^3}\right] + C_1$$

B.C. 1 $v'(L) = 0 \qquad \therefore C_1 = -\frac{PL^2}{12EI_A}$

$$v' = \frac{PL^4}{EI_A}\left[\frac{L+3x}{6(L+x)^3}\right] - \frac{PL^2}{12EI_A}$$

or

$$v' = \frac{PL^4}{EI_A}\left[\frac{L}{6(L+x)^3}\right] + \frac{PL^4}{EI_A}\left[\frac{x}{2(L+x)^3}\right] - \frac{PL^2}{12EI_A} \qquad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $$\int \frac{dx}{(L+x)^3} = -\frac{1}{2(L+x)^2}$$

$$\int \frac{xdx}{(L+x)^3} = \frac{-(L+2x)}{2(L+x)^2}$$

$$v = \frac{PL^4}{EI_A}\left(\frac{L}{6}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{L+x}\right)^2 + \frac{PL^4}{EI_A}\left(\frac{1}{2}\right)\left[-\frac{L+2x}{2(L+x)^2}\right]$$

$$-\frac{PL^2}{12EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[-\frac{L^2}{12(L+x)^2} - \frac{L(L+2x)}{4(L+x)^2} - \frac{x}{12L}\right] + C_2 \quad (3)$$

B.C. 2 $v(L) = 0$ $\quad \therefore C_2 = \frac{PL^3}{EI_A}\left(\frac{7}{24}\right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3)

$$v = \frac{PL^3}{24EI_A}\left[7 - \frac{4L(2L+3x)}{(L+x)^2} - \frac{2x}{L}\right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{24EI_A} \quad \text{(positive downward)} \quad \leftarrow$$

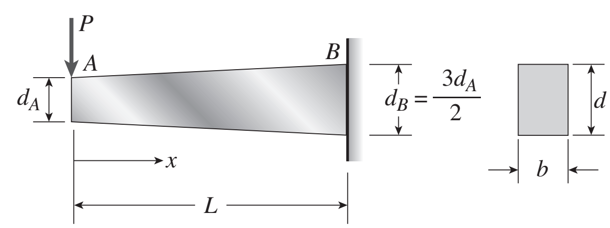
Problem 9.7-9 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular with constant width b, depth d_A at support A, and depth $d_B = 3d_A/2$ at the support. Thus, the depth d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{2L}(2L + x)$$

$$I = \frac{bd^3}{12} = \frac{bd_A^3}{96L^3}(2L+x)^3 = \frac{I_A}{8L^3}(2L+x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-9 Tapered cantilever beam

$M = -Px \qquad EIv'' = -Px \qquad I = \frac{I_A}{8L^3}(2L+x)^3$

$$v'' = -\frac{Px}{EI} = -\frac{8PL^3}{EI_A}\left[\frac{x}{(2L+x)^3}\right] \quad (1)$$

INTEGRATE EQ. (1)

From Appendix C: $$\int \frac{xdx}{(2L+x)^3} = -\frac{2L+2x}{2(2L+x)^2}$$

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L+x}{(2L+x)^3}\right] + C_1$$

B.C. 1 $v'(L) = 0$ $\quad \therefore C_1 = -\frac{16PL^2}{9EI_A}$

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L+x}{(2L+x)^2}\right] - \frac{16PL^2}{9EI_A}$$

or

$$v' = \frac{8PL^3}{EI_A}\left[\frac{L}{(2L+x)^2}\right] + \frac{8PL^3}{EI_A}\left[\frac{x}{(2L+x)^2}\right]$$

$$-\frac{16PL^2}{9EI_A} \quad (2)$$

INTEGRATE EQ. (2)

From Appendix C: $\displaystyle\int \frac{dx}{(2L+x)^2} = -\frac{1}{2L+x}$

$$\int \frac{x\,dx}{(2L+x)^2} = \frac{2L}{2L+x} + \ln(2L+x)$$

$$v = \frac{8PL^3}{EI_A}\left(-\frac{L}{2L+x}\right) + \frac{8PL^3}{EI_A}\left[\frac{2L}{2L+x} + \ln(2L+x)\right] - \frac{16PL^2}{9EI_A}x + C_2$$

$$= \frac{PL^3}{EI_A}\left[\frac{8L}{2L+x} + 8\ln(2L+x) - \frac{16x}{9L}\right] + C_2 \qquad (3)$$

B.C. 2 $v(L) = 0$ $\qquad \therefore C_2 = -\dfrac{8PL^3}{EI_A}\left[\dfrac{1}{9} + \ln(3L)\right]$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{8PL^3}{EI_A}\left[\frac{L}{2L+x} - \frac{2x}{9L} - \frac{1}{9} + \ln\left(\frac{2L+x}{3L}\right)\right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

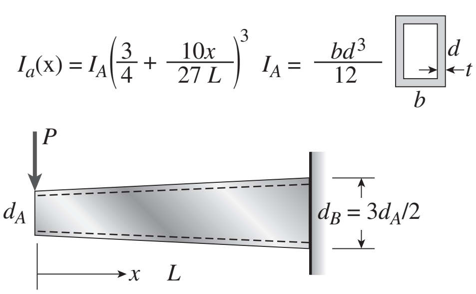
$$\delta_A = -v(0) = \frac{8PL^2}{EI_A}\left[\ln\left(\frac{3}{2}\right) - \frac{7}{18}\right]$$

$$= 0.1326\frac{PL^3}{EI_A} \qquad \text{(positive downward)} \quad \leftarrow$$

NOTE: $\ln\dfrac{2}{3} = -\ln\dfrac{3}{2}$

Problem 9.7-10 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular tubes with constant width b and *outer tube* depth d_A at A, and *outer tube* depth $d_B = 3d_A/2$ at support B. The tube thickness is constant, $t = d_A/20$. I_A is the moment of inertia of the *outer tube* at end A of the beam.

If the moment of inertia of the tube is approximated as $I_a(x)$ as defined, find the *equation* of the deflection curve and the deflection δ_A at the free end of the beam due to the load P.



Solution 9.7-10

BENDING-MOMENT EQUATIONS

$EIv'' = M = -Px$

$$v'' = \frac{-Px}{EI_a(x)} = \frac{-Px}{EI_A\left(\frac{3}{4} + \frac{10x}{27L}\right)^3} = \frac{-P}{EI_A}\,\frac{x}{\left(\frac{3}{4} + \frac{10x}{27L}\right)^3}$$

From Appendix C: $\displaystyle\int \frac{x}{(a+bx)^3}\,dx = -\frac{a+2bx}{2b^2(a+bx)^2}$

$$v' = \frac{-P}{EI_A}\left[\frac{\frac{3}{4} + 2\frac{10}{27L}x}{2\left(\frac{10}{27L}\right)^2\left(\frac{3}{4} + \frac{10}{27L}x\right)^2}\right] + C_1$$

$$v' = \frac{PL^3}{EI_A}\,\frac{19683}{50}\left[\frac{81L}{(81L+40x)^2} + \frac{80x}{(81L+40x)^2}\right] + C_1$$

From Appendix C: $\displaystyle\int \frac{1}{(a+bx)^2}\,dx = \frac{-1}{b(a+bx)}$

$$\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a+bx} + ln(a+bx)\right)$$

$$v = \frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{-81L}{40(81L+40x)} + \frac{80}{40^2}\left(\frac{81L}{81L+40x} + \ln(81L+40x)\right)\right] + C_1x + C_2$$

$$v = \frac{19683PL^3}{2000EI_A}\left(\frac{81L + 162\ln(81L+40x)L + 80\ln(81L+40x)x}{81L+40x}\right) + C_1x + C_2$$

B.C. $v'(L) = 0$ $\qquad C_1 = \dfrac{-3168963}{732050}\dfrac{PL^2}{EI_A}$

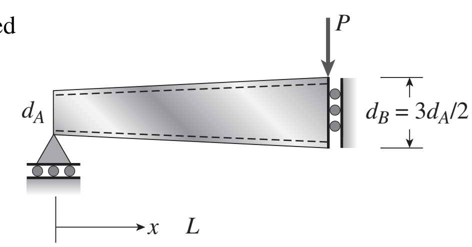
B.C. $v(L) = 0$ $\qquad C_2 = \dfrac{-19683PL^3}{29282000EI_A}(3361 + 29282\ln(121L))$

$$v(x) = \frac{19683PL^3}{2000EI_A}\left(\frac{81L}{81L+40x} + 2\ln\left(\frac{81}{121} + \frac{40x}{121L}\right) - \frac{6440x}{14641L} - \frac{3361}{14641}\right) \quad \leftarrow$$

$$\delta_A = -v(0) = \frac{19683PL^3}{7320500EI_A}\left(-2820 + 14641\ln\left(\frac{11}{9}\right)\right) = 0.317\frac{PL^3}{EI_A} \quad \leftarrow$$

****Problem 9.7-11** Repeat Problem 9.7-10 but now use the tapered propped cantilever tube AB, with guided support at B, shown in the figure which supports a concentrated load P at the guided end.

Find the equation of the deflection curve and the deflection δ_B at the guided end of the beam due to the load P.



Solution 9.7-11

Bending-moment equations

$EIv'' = M = Px$

$$v'' = \frac{Px}{EI_a(x)} = \frac{Px}{EI_A\left(\frac{3}{4} + \frac{10x}{27L}\right)^3} = \frac{P}{EI_A}\frac{x}{\left(\frac{3}{4} + \frac{10x}{27L}\right)^3}$$

From Appendix C: $\displaystyle\int \frac{x}{(a+bx)^3}\,dx = -\frac{a+2bx}{2b^2(a+bx)^2}$

$$v' = \frac{P}{EI_A}\left[\frac{\frac{3}{4} + 2\frac{10}{27L}x}{2\left(\frac{10}{27L}\right)^2\left(\frac{3}{4} + \frac{10}{27L}x\right)^2}\right] + C_1$$

$$\nu' = -\frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{81L}{(81L+40x)^2}+\frac{80x}{(81L+40x)^2}\right]+C_1$$

From Appendix C: $\displaystyle\int \frac{1}{(a+bx)^2}\,dx = \frac{-1}{b(a+bx)}$

$$\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a+bx}+\ln(a+bx)\right)$$

$$\nu = -\frac{PL^3}{EI_A}\frac{19683}{50}\left[\frac{-81L}{40(81L+40x)}+\frac{80}{40^2}\left(\frac{81L}{81L+40x}+\ln(81L+40x)\right)\right]+C_1x+C_2$$

$$\nu = -\frac{19683PL^3}{2000EI_A}\left(\frac{81L+162\ln(81L+40x)L+80\ln(81L+40x)x}{81L+40x}\right)+C_1x+C_2$$

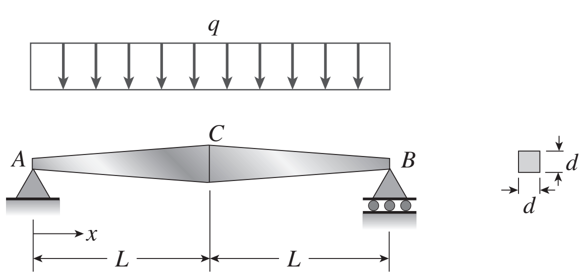
B.C. $\nu'(L) = 0$ $\qquad C_1 = \dfrac{3168963}{732050}\dfrac{PL^2}{EI_A}$

B.C. $\nu(L) = 0$ $\qquad C_2 = \dfrac{19683PL^3}{2000EI_A}(1+2\ln(81L))$

$$\nu(x) = -\frac{19683PL^3}{2000EI_A}\left(\frac{81L}{81L+40x}+2\ln\left(1+\frac{40x}{81L}\right)-\frac{6440x}{14641L}-1\right) \quad \leftarrow$$

$$\delta_B = -\nu(L) = \frac{19683PL^3}{7320500EI_A}\left(-2820+14641\ln\left(\frac{11}{9}\right)\right) = 0.317\frac{PL^3}{EI_A} \quad \leftarrow$$

Problem 9.7-12 A simple beam ACB is constructed with square cross sections and a double taper (see figure). The depth of the beam at the supports is d_A and at the midpoint is $d_C = 2d_A$. Each half of the beam has length L. Thus, the depth d and moment of inertia I at distance x from the left-hand end are, respectively,



$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{d^4}{12} = \frac{d_A^4}{12L^4}(L + x)^4 = \frac{I_A}{L^4}(L + x)^4$$

in which I_A is the moment of inertia at end A of the beam. (These equations are valid for x between 0 and L, that is, for the left-hand half of the beam.)

(a) Obtain equations for the slope and deflection of the left-hand half of the beam due to the uniform load.

(b) From those equations obtain formulas for the angle of rotation θ_A at support A and the deflection δ_C at the midpoint.

Solution 9.7-12 Simple beam with a double taper

L = length of one-half of the beam

$$I = \frac{I_A}{L^4}(L+x)^4 \qquad (0 \le x \le L)$$

(x is measured from the left-hand support A)

Reactions: $R_A = R_B = qL$

Bending moment: $M = R_A x - \dfrac{qx^2}{2} = qLx - \dfrac{qx^2}{2}$

From Eq. (9-12a):

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$v'' = \frac{qL^5 x}{EI_A(L+x)^4} - \frac{qL^4x^2}{2EI_A(L+x)^4} \qquad (0 \le x \le L) \quad (1)$$

Integrate Eq. (1)

From Appendix C: $\displaystyle\int \frac{x\,dx}{(L+x)^4} = -\frac{L+3x}{6(L+x)^3}$

$$\int \frac{x^2dx}{(L+x)^4} = -\frac{L^2+3Lx+3x^2}{3(L+x)^3}$$

$$v' = \frac{qL^5}{EI_A}\left[-\frac{L+3x}{6(L+x)^3}\right] - \frac{qL^4}{2EI_A}\left[-\frac{L^2+3Lx+3x^2}{3(L+x)^3}\right] + C_1$$

$$= \frac{qL^4x^2}{2EI_A(L+x)^3} + C_1 \qquad (0 \le x \le L) \qquad (2)$$

B.C. 1 (symmetry) $v'(L) = 0 \qquad \therefore C_1 = -\dfrac{qL^3}{16EI_A}$

Slope of the beam

Substitute C_1 into Eq. (2).

$$v' = \frac{qL^4x^2}{2EI_A(L+x)^3} - \frac{qL^3}{16EI_A}$$

$$= -\frac{qL^3}{16EI_A}\left[1 - \frac{8Lx^2}{(L+x)^3}\right] \qquad (0 \le x \le L) \qquad (3)$$

Angle of rotation at support A

$$\theta_A = -v'(0) = \frac{qL^3}{16EI_A} \qquad \text{(positive clockwise)} \qquad \leftarrow$$

Integrate Eq. (3)

From Appendix C: $\displaystyle\int \frac{x^2dx}{(L+x)^3} = \frac{L(3L+4x)}{2(L+x)^2} + \ln(L+x)$

$$v = -\frac{qL^3}{16EI_A}\left[x - \frac{8L^2(3L+4x)}{2(L+x)^2} - 8L\ln(L+x)\right] + C_2 \qquad (0 \le x \le L) \qquad (4)$$

B.C. 2 $v(0) = 0 \qquad \therefore C_2 = -\dfrac{qL^4}{2EI_A}\left(\dfrac{3}{2} + \ln L\right)$

Deflection of the beam

Substitute C_2 into Eq. (4) and simplify. (The algebra is lengthy.)

$$v = -\frac{qL^4}{2EI_A}\left[\frac{(9L^2+14Lx+x^2)x}{8L(L+x)^2} - \ln\left(1+\frac{x}{L}\right)\right]$$

$$(0 \le x \le L) \qquad \leftarrow$$

Deflection at the midpoint C of the beam

$$\delta_C = -v(L) = \frac{qL^4}{8EI_A}(3 - 4\ln 2) = 0.02843\frac{qL^4}{EI_A}$$

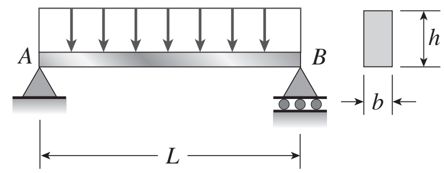
(positive downward) $\leftarrow$

Strain Energy

The beams described in the problems for Section 9.8 have constant flexural rigidity EI.

Problem 9.8-1 A uniformly loaded simple beam AB (see figure) of span length L and rectangular cross section (b = width, h = height) has a maximum bending stress σ_{max} due to the uniform load.

Determine the strain energy U stored in the beam.



Solution 9.8-1 Simple beam with a uniform load

Given: L, b, h, σ_{max} Find: U (strain energy)

Bending moment: $M = \dfrac{qLx}{2} - \dfrac{qx^2}{2}$

Strain energy (Eq. 9-80a): $U = \displaystyle\int_0^L \frac{M^2 dx}{2EI}$

$$= \frac{q^2L^5}{240EI} \qquad (1)$$

Maximum stress: $\sigma_{max} = \dfrac{M_{max}c}{I} = \dfrac{M_{max}h}{2I}$

$M_{max} = \dfrac{qL^2}{8}$ $\sigma_{max} = \dfrac{qL^2h}{16I}$

Solve for q: $q = \dfrac{16I\sigma_{max}}{L^2h}$

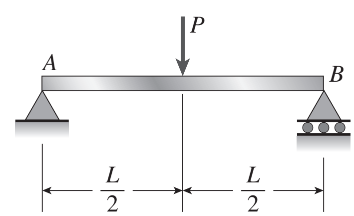
Substitute q into Eq. (1):

$$U = \frac{16I\sigma_{max}^2 L}{15h^2E}$$

Substitute $I = \dfrac{bh^3}{12}$: $U = \dfrac{4bhL\sigma_{max}^2}{45E}$ ←

Problem 9.8-2 A simple beam AB of length L supports a concentrated load P at the midpoint (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.
(b) Evaluate the strain energy of the beam from the equation of the deflection curve.
(c) From the strain energy, determine the deflection δ under the load P.



Solution 9.8-2 Simple beam with a concentrated load

(a) Bending moment $M = \dfrac{Px}{2} \quad \left(0 \le x \le \dfrac{L}{2}\right)$

Strain energy (Eq. 9-80a):

$$U = 2\int_0^{L/2} \frac{M^2 dx}{2EI} = \frac{P^2L^3}{96EI} \quad ←$$

(b) Deflection curve

From Table G-2, Case 4:

$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{dv}{dx} = -\frac{P}{16EI}(L^2 - 4x^2) \qquad \frac{d^2v}{dx^2} = \frac{Px}{2EI}$$

Strain energy (Eq. 9-80b):

$$U = 2\int_0^{L/2} \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = EI\int_0^{L/2}\left(\frac{Px}{2EI}\right)^2 dx$$

$$= \frac{P^2L^3}{96EI} \quad \leftarrow$$

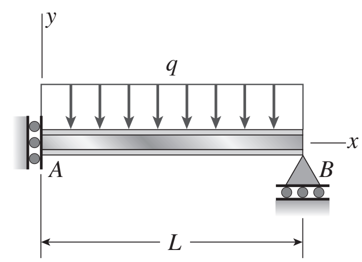
(c) Deflection δ under the load P

From Eq. (9-82a):

$$\delta = \frac{2U}{P} = \frac{PL^3}{48EI} \quad \leftarrow$$

Problem 9.8-3 A propped cantilever beam AB of length L, and with guided support at A, supports a uniform load of intensity q (see figure).

(a) Evaluate the strain energy of the beam from the bending moment in the beam.
(b) Evaluate the strain energy of the beam from the equation of the deflection curve.



Solution 9.8-3

(a) Bending-moment equations

Measure x from end B

$$M = qLx - \frac{qx^2}{2}$$

Strain Energy (Eq. 9-80a):

$$U = \int_0^L \frac{M^2}{2EI}dx$$

$$= \int_0^L \frac{1}{2EI}\left(qLx - \frac{qx^2}{2}\right)^2 dx = \frac{q^2L^5}{15EI} \quad \leftarrow$$

(b) Deflection curve

Measure x from end B

$$v = -\frac{qx}{24EI}(8L^3 - 4Lx^2 + x^3)$$

$$\frac{d}{dx}v = -\frac{q}{24EI}(8L^3 - 12Lx^2 + 4x^3)$$

$$\frac{d^2}{dx^2}v = -\frac{q}{2EI}(-2Lx + x^2)$$

Strain energy (Eq. 9-80b):

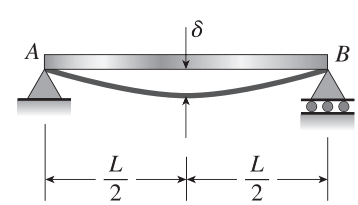
$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2}{dx^2}v\right)^2 dx$$

$$U = \int_0^L \frac{EI}{2}\left[-\frac{q}{2EI}(-2Lx + x^2)\right]^2 dx$$

$$U = \frac{q^2L^5}{15EI} \quad \leftarrow$$

Problem 9.8-4 A simple beam AB of length L is subjected to loads that produce a symmetric deflection curve with maximum deflection δ at the midpoint of the span (see figure).

How much strain energy U is stored in the beam if the deflection curve is (a) a parabola, and (b) a half wave of a sine curve?



Solution 9.8-4 Simple beam (symmetric deflection curve)

GIVEN: L, EI, δ δ = maximum deflection at midpoint

Determine the strain energy U.

Assume the deflection ν is positive downward.

(a) DEFLECTION CURVE IS A PARABOLA

$$\nu = \frac{4\delta x}{L^2}(L-x) \qquad \frac{d\nu}{dx} = \frac{4\delta}{L^2}(L-2x)$$

$$\frac{d^2\nu}{dx^2} = -\frac{8\delta}{L^2}$$

Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2\nu}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(-\frac{8\delta}{L^2}\right)^2 dx$$

$$= \frac{32EI\delta^2}{L^3} \quad \leftarrow$$

(b) DEFLECTION CURVE IS A SINE CURVE

$$\nu = \delta \sin\frac{\pi x}{L} \qquad \frac{d\nu}{dx} = \frac{\pi\delta}{L}\cos\frac{\pi x}{L}$$

$$\frac{d^2\nu}{dx^2} = -\frac{\pi^2\delta}{L^2}\sin\frac{\pi x}{L}$$

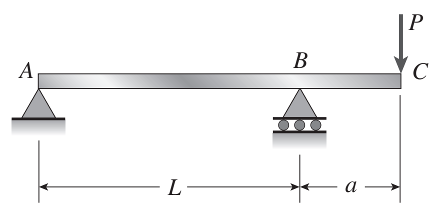
Strain energy (Eq. 9-80b):

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2\nu}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L \left(-\frac{\pi^2\delta}{L^2}\right)^2 \sin^2\frac{\pi x}{L}\, dx$$

$$= \frac{\pi^4 EI\delta^2}{4L^3} \quad \leftarrow$$

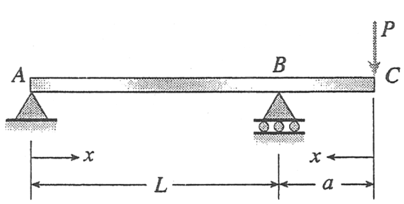
Problem 9.8-5 A beam ABC with simple supports at A and B and an overhang BC supports a concentrated load P at the free end C (see figure).

(a) Determine the strain energy U stored in the beam due to the load P.
(b) From the strain energy, find the deflection δ_C under the load P.
(c) Calculate the numerical values of U and δ_C if the length L is 8 ft, the overhang length a is 3 ft, the beam is a W 10 × 12 steel wide-flange section, and the load P produces a maximum stress of 12,000 psi in the beam. (Use $E = 29 \times 10^6$ psi.)



Solution 9.8-5 Simple beam with an overhang

(a) STRAIN ENERGY (use Eq. 9-80a)



FROM A TO B: $M = -\dfrac{Pax}{L}$

$$U_{AB} = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{1}{2EI}\left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

FROM B TO C: $M = -Px$

$$U_{BC} = \int_0^a \frac{1}{2EI}(-Px)^2\, dx = \frac{P^2a^3}{6EI}$$

TOTAL STRAIN ENERGY:

$$U = U_{AB} + U_{BC} = \frac{P^2a^2}{6EI}(L + a) \quad \leftarrow$$

(b) DEFLECTION δ_C UNDER THE LOAD P

From Eq. (9-82a):

$$\delta_C = \frac{2U}{P} = \frac{Pa^2}{3EI}(L + a) \quad \leftarrow$$

(c) CALCULATE U AND δ_c

Data: $L = 8$ ft $= 96$ in. $a = 3$ ft $= 36$ in.

W 10×12 $E = 29 \times 10^6$ psi

$\sigma_{max} = 12{,}000$ psi

$I = 53.8$ in.^{4} $c = \dfrac{d}{2} = \dfrac{9.87}{2} = 4.935$ in.

Express load P in terms of maximum stress:

$$\sigma_{max} = \frac{Mc}{I} = \frac{M_{max}c}{I} = \frac{Pac}{I} \quad \therefore P = \frac{\sigma_{max} I}{ac}$$

$$U = \frac{P^2a^2(L+a)}{6EI} = \frac{\sigma_{max}^2 I(L+a)}{6c^2E}$$

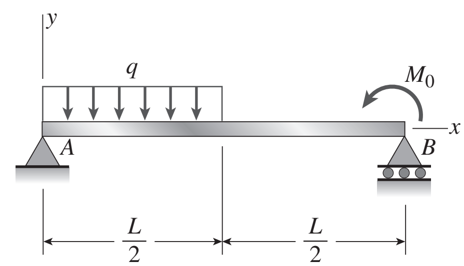
$$= 241 \text{ in.-lb} \quad \leftarrow$$

$$\delta_c = \frac{Pa^2(L+a)}{3EI} = \frac{\sigma_{max} a(L+a)}{3cE}$$

$$= 0.133 \text{ in.} \quad \leftarrow$$

Problem 9.8-6 A simple beam ACB supporting a uniform load q over the first half of the beam and a couple of moment M_0 at end B is shown in the figure.

Determine the strain energy U stored in the beam due to the load q and the couple M_0 acting simultaneously.



Solution 9.8-6

FROM A TO MID-SPAN

Bending-Moment Equations

$$M = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}$$

STRAIN ENERGY (EQ. 9-80A):

$$U_1 = \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx$$

$$= \int_0^{\frac{L}{2}} \frac{1}{2EI}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}\right]^2 dx$$

$$U_1 = \frac{L}{3840EI}\left(3L^4q^2 + 30qL^2M_0 + 80M_0^2\right)$$

FROM MID-SPAN TO B

Bending-Moment Equations

$$M = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qL}{2}\left(x - \frac{L}{4}\right)$$

STRAIN ENERGY (EQ. 9-80A):

$$U_2 = \int_{\frac{L}{2}}^{L} \frac{M^2}{2EI}\,dx = \int_{\frac{L}{2}}^{L} \frac{1}{2EI}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qL}{2}\left(x - \frac{L}{4}\right)\right]^2 dx$$

$$U_2 = \frac{L}{3072EI}\left(L^4q^2 + 32qL^2M_0 + 448M_0^2\right)$$

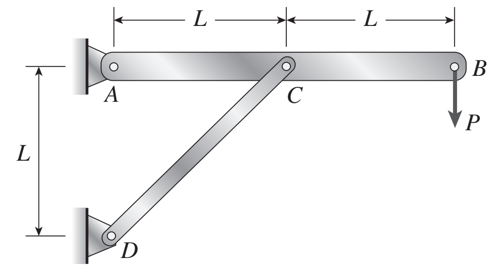
STRAIN ENERGY OF THE ENTIRE BEAM

$$U = U_1 + U_2 = \frac{L}{15360EI}\left(17L^4q^2 + 280qL^2M_0 + 2560M_0^2\right) \quad \leftarrow$$

Problem 9.8-7 The frame shown in the figure consists of a beam ACB supported by a struct CD. The beam has length $2L$ and is continuous through joint C. A concentrated load P acts at the free end B.

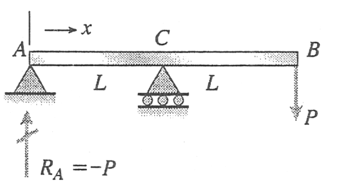
Determine the vertical deflection δ_B at point B due to the load P.

Note: Let EI denote the flexural rigidity of the beam, and let EA denote the axial rigidity of the strut. Disregard axial and shearing effects in the beam, and disregard any bending effects in the strut.



Solution 9.8-7 Frame with beam and strut

BEAM ACB



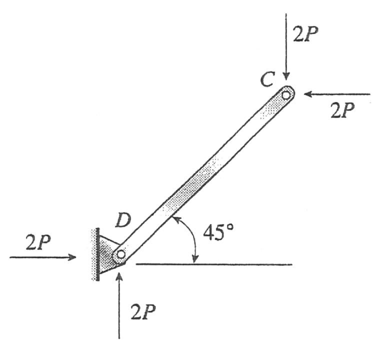
For part AC of the beam: $M = -Px$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L (-Px)^2 dx = \frac{P^2L^3}{6EI}$$

For part CB of the beam: $U_{CB} = U_{AC} = \dfrac{P^2L^3}{6EI}$

Entire beam: $U_{\text{BEAM}} = U_{AC} + U_{CB} = \dfrac{P^2L^3}{3EI}$

SRUCT CD



L_{CD} = length of strut

$= \sqrt{2}L$

F = axial force in strut

$= 2\sqrt{2}P$

$$U_{\text{STRUT}} = \frac{F^2L_{CD}}{2EA} \qquad \text{(Eq. 2-37a)}$$

$$U_{\text{STRUT}} = \frac{(2\sqrt{2}P)^2(\sqrt{2}L)}{2EA} = \frac{4\sqrt{2}P^2L}{EA}$$

FRAME $\quad U = U_{\text{BEAM}} + U_{\text{STRUT}} = \dfrac{P^2L^3}{3EI} + \dfrac{4\sqrt{2}P^2L}{EA}$

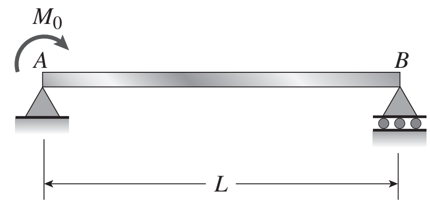
DEFLECTION δ_B AT POINT B

From Eq. (9-82a):

$$\delta_B = \frac{2U}{P} = \frac{2PL^3}{3EI} + \frac{8\sqrt{2}PL}{EA} \qquad \leftarrow$$

Castigliano's Theorem

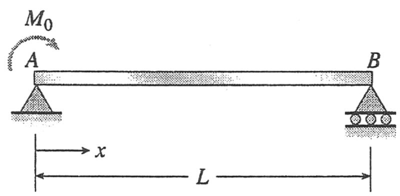
The beams described in the problems for Section 9.9 have constant flexural rigidity EI.



Problem 9.9-1 A simple beam AB of length L is loaded at the left-hand end by a couple of moment M_0 (see figure).

Determine the angle of rotation θ_A at support A. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)

Solution 9.9-1 Simple beam with couple M_0



$$R_A = \frac{M_0}{L} \quad \text{(downward)}$$

$$M = M_0 - R_A x = M_0 - \frac{M_0 x}{L}$$
$$= M_0\left(1 - \frac{x}{L}\right)$$

Strain energy

$$U = \int \frac{M^2 dx}{2EI} = \frac{M_0^2}{2EI}\int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \frac{M_0^2 L}{6EI}$$

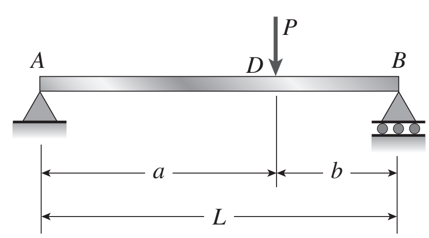
Castigliano's theorem

$$\theta_A = \frac{dU}{dM_0} = \frac{M_0 L}{3EI} \quad \text{(clockwise)} \qquad \leftarrow$$

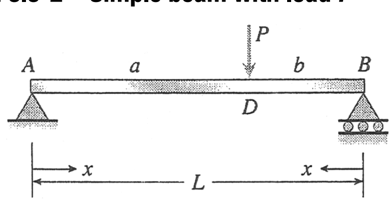
(This result agrees with Case 7, Table G-2)

Problem 9.9-2 The simple beam shown in the figure supports a concentrated load P acting at distance a from the left-hand support and distance b from the right-hand support.

Determine the deflection δ_D at point D where the load is applied. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-2 Simple beam with load P



$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

$$M_{AD} = R_A x = \frac{Pbx}{L}$$

$$M_{DB} = R_B x = \frac{Pax}{L}$$

Strain energy $\quad U = \int \frac{M^2 dx}{2EI}$

$$U_{AD} = \frac{1}{2EI}\int_0^a \left(\frac{Pbx}{L}\right)^2 dx = \frac{P^2 a^3 b^2}{6EIL^2}$$

$$U_{DB} = \frac{1}{2EI}\int_0^b \left(\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2b^3}{6EIL^2}$$

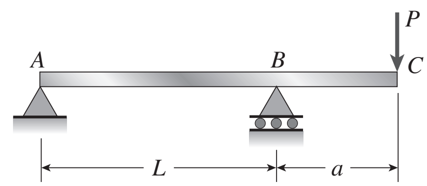
$$U = U_{AD} + U_{DB} = \frac{P^2a^2b^2}{6LEI}$$

CASTIGLIANO'S THEOREM

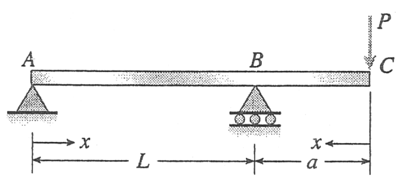
$$\delta_D = \frac{dU}{dP} = \frac{Pa^2b^2}{3LEI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-3 An overhanging beam ABC supports a concentrated load P at the end of the overhang (see figure). Span AB has length L and the overhang has length a.

Determine the deflection δ_C at the end of the overhang. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-3 Overhanging beam



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$M_{AB} = -R_Ax = -\frac{Pax}{L}$$

$$M_{CB} = -Px$$

STRAIN ENERGY $U = \int \frac{M^2dx}{2EI}$

$$U_{AB} = \frac{1}{2EI}\int_0^L \left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

$$U_{CB} = \frac{1}{2EI}\int_0^a (-Px)^2 dx = \frac{P^2a^3}{6EI}$$

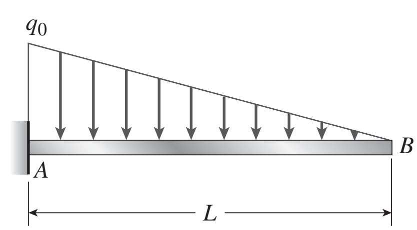
$$U = U_{AB} + U_{CB} = \frac{P^2a^2}{6EI}(L + a)$$

CASTIGLIANO'S THEOREM

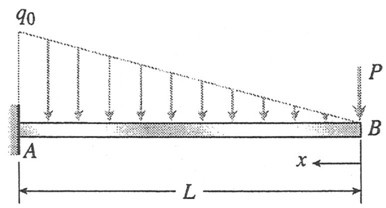
$$\delta_C = \frac{dU}{dP} = \frac{Pa^2}{3EI}(L + a) \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-4 The cantilever beam shown in the figure supports a triangularly distributed load of maximum intensity q_0.

Determine the deflection δ_B at the free end B. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-4 Cantilever beam with triangular load



$P =$ fictitious load corresponding to deflection δ_B

$$M = -Px - \frac{q_0 x^3}{6L}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L \left(-Px - \frac{q_0 x^3}{6L}\right)^2 dx$$

$$= \frac{P^2L^3}{6EI} + \frac{Pq_0L^4}{30EI} + \frac{q_0^2L^5}{42EI}$$

CASTIGLIANO'S THEOREM

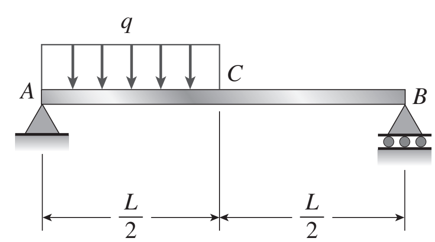
$$\delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{q_0L^4}{30EI} \quad \text{(downward)}$$

(This result agrees with Cases 1 and 8 of Table G-1.)

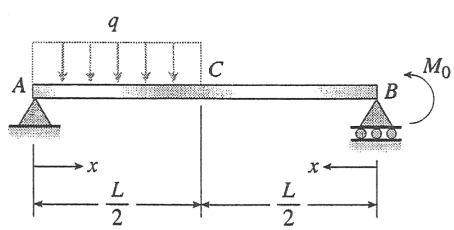
SET $P = 0$: $\delta_B = \dfrac{q_0L^4}{30EI}$ $\leftarrow$

Problem 9.9-5 A simple beam *ACB* supports a uniform load of intensity q on the left-hand half of the span (see figure).

Determine the angle of rotation θ_B at support B. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-5 Simple beam with partial uniform load



$M_0 =$ fictitious load corresponding to angle of rotation θ_B

$$R_A = \frac{3qL}{8} + \frac{M_0}{L} \qquad R_B = \frac{qL}{8} - \frac{M_0}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT *AC*

$$M_{AC} = R_A x - \frac{qx^2}{2} = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}$$

$$\left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = \frac{x}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT *CB*

$$M_{CB} = R_B x + M_0 = \left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0$$

$$\left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -\frac{x}{L} + 1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

$$\theta_B = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}\right]\left[\frac{x}{L}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0\right]\left[1 - \frac{x}{L}\right]dx$$

SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\theta_B = \frac{1}{EI}\int_0^{L/2}\left(\frac{3qLx}{8} - \frac{qx^2}{2}\right)\left(\frac{x}{L}\right)dx$$

$$+ \frac{1}{EI}\int_0^{L/2}\left(\frac{qLx}{8}\right)\left(1 - \frac{x}{L}\right)dx$$

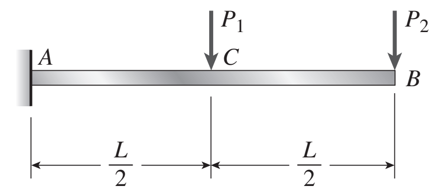
$$= \frac{qL^3}{128EI} + \frac{qL^3}{96EI}$$

$$= \frac{7qL^3}{384EI} \quad \text{(counterclockwise)} \quad \leftarrow$$

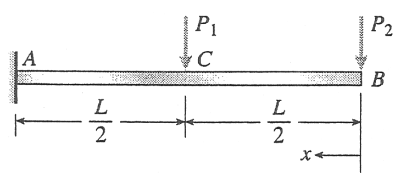
(This result agrees with Case 2, Table G-2.)

Problem 9.9-6 A cantilever beam *ACB* supports two concentrated loads P_1 and P_2, as shown in the figure.

Determine the deflections δ_C and δ_B at points *C* and *B*, respectively. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-6 Cantilever beam with loads P_1 and P_2



BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT *CB*

$$M_{CB} = -P_2x \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial P_1} = 0 \qquad \frac{\partial M_{CB}}{\partial P_2} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT *AC*

$$M_{AC} = -P_1\left(x - \frac{L}{2}\right) - P_2x \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{AC}}{\partial P_1} = \frac{L}{2} - x \qquad \frac{\partial M_{AC}}{\partial P_2} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\delta_C = \frac{1}{EI}\int_0^{L/2}(M_{CB})\left(\frac{\partial M_{CB}}{\partial P_1}\right)dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}(M_{AC})\left(\frac{\partial M_{AC}}{\partial P_1}\right)dx$$

$$= 0 + \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2x\right]\left(\frac{L}{2} - x\right)dx$$

$$= \frac{L^3}{48EI}(2P_1 + 5P_2) \quad \leftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_B

$$\delta_B = \frac{1}{EI}\int_0^{L/2}(M_{CB})\left(\frac{\partial M_{CB}}{\partial P_2}\right)dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}(M_{AC})\left(\frac{\partial M_{AC}}{\partial P_2}\right)dx$$

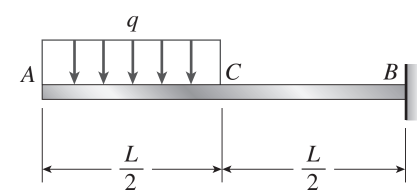
$$= \frac{1}{EI}\int_0^{L/2}(-P_2x)(-x)dx$$
$$+ \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2x\right](-x)dx$$

$$= \frac{P_2L^3}{24EI} + \frac{L^3}{48EI}(5P_1 + 14P_2)$$
$$= \frac{L^3}{48EI}(5P_1 + 16P_2) \quad \leftarrow$$

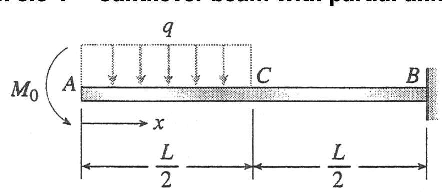
(These results can be verified with the aid of Cases 4 and 5, Table G-1.)

Problem 9.9-7 The cantilever beam ACB shown in the figure is subjected to a uniform load of intensity q acting between points A and C.

Determine the angle of rotation θ_A at the free end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-7 Cantilever beam with partial uniform load



M_0 = fictitious load corresponding to the angle of rotation θ_A

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = -M_0 - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = -1$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = -M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

$$\theta_A = \int\left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$
$$= \frac{1}{EI}\int_0^{L/2}\left(-M_0 - \frac{qx^2}{2}\right)(-1)dx$$
$$+ \frac{1}{EI}\int_{L/2}^{L}\left[-M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right)\right](-1)dx$$

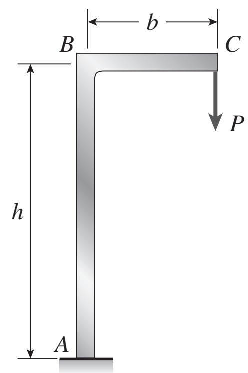
SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\theta_A = \frac{1}{EI}\int_0^{L/2}\frac{qx^2}{2}dx + \frac{1}{EI}\int_{L/2}^{L}\left(\frac{qL}{2}\right)\left(x - \frac{L}{4}\right)dx$$
$$= \frac{qL^3}{48EI} + \frac{qL^3}{8EI}$$
$$= \frac{7qL^3}{48EI} \quad \text{(counterclockwise)} \quad \leftarrow$$

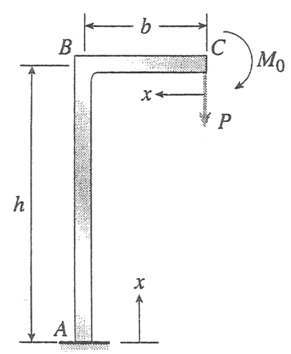
(This result can be verified with the aid of Case 3, Table G-1.)

Problem 9.9-8 The frame ABC supports a concentrated load P at point C (see figure). Members AB and BC have lengths h and b, respectively.

Determine the vertical deflection δ_C and angle of rotation θ_C at end C of the frame. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-8 Frame with concentrated load



P = concentrated load acting at point C (corresponding to the deflection δ_C)

M_0 = fictitious moment corresponding to the angle of rotation θ_C

BENDING MOMENT AND PARTIAL DERIVATIVES FOR MEMBER AB

$$M_{AB} = Pb + M_0 \quad (0 \le x \le h)$$

$$\frac{\partial M_{AB}}{\partial P} = b \qquad \frac{\partial M_{AB}}{M_0} = 1$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR MEMBER BC

$$M_{BC} = Px + M_0 \quad (0 \le x \le b)$$

$$\frac{\partial M_{BC}}{\partial P} = x \qquad \frac{\partial M_{BC}}{\partial M_0} = 1$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(b)\,dx + \frac{1}{EI}\int_0^b (Px + M_0)(x)\,dx$$

Set $M_0 = 0$:

$$\delta_C = \frac{1}{EI}\int_0^h Pb^2 dx + \frac{1}{EI}\int_0^b Px^2 dx$$

$$= \frac{Pb^2}{3EI}(3h + b) \quad \text{(downward)} \quad \leftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_C

$$\theta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(1) dx + \frac{1}{EI}\int_0^b (Px + M_0)(1)\,dx$$

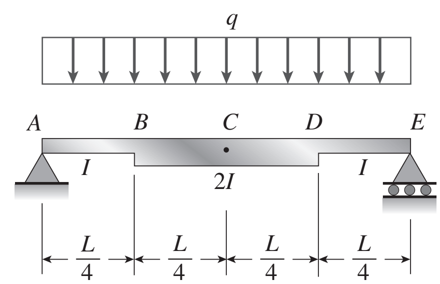
Set $M_0 = 0$:

$$\theta_C = \frac{1}{EI}\int_0^h Pb\,dx + \frac{1}{EI}\int_0^b Px dx$$

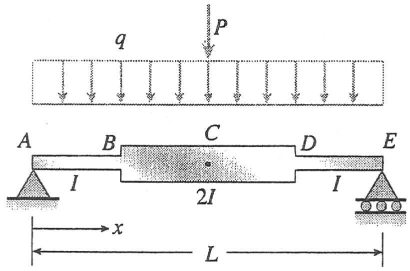
$$= \frac{Pb}{2EI}(2h + b) \quad \text{(clockwise)} \quad \leftarrow$$

Problem 9.9-9 A simple beam *ABCDE* supports a uniform load of intensity q (see figure). The moment of inertia in the central part of the beam (BCD) is twice the moment of inertia in the end parts (AB and DE).

Find the deflection δ_C at the midpoint C of the beam. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-9 Nonprismatic beam



P = fictitiuous load corresponding to the deflection δ_C at the midpoint

$$R_A = \frac{qL}{2} + \frac{P}{2}$$

Bending moment and partial derivative for the left-hand half of the beam (A to C)

$$M_{AC} = \frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial P} = \frac{x}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

Modified Castigliano's theorem (Eq. 9-88)

Integrate from A to C and multiply by 2.

$$\delta_C = 2\int \left(\frac{M_{AC}}{EI}\right)\left(\frac{\partial M_{AC}}{\partial P}\right)dx$$

$$= 2\left(\frac{1}{EI}\right)\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

$$+ 2\left(\frac{1}{2EI}\right)\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

Set fictitious load P equal to zero

$$\delta_C = \frac{2}{EI}\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

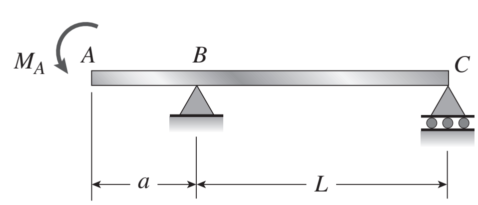
$$+ \frac{1}{EI}\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

$$= \frac{13qL^4}{6{,}144EI} + \frac{67qL^4}{12{,}288EI}$$

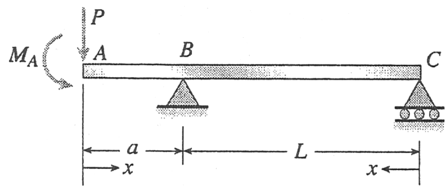
$$\delta_C = \frac{31qL^4}{4096EI} \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-10 An overhanging beam *ABC* is subjected to a couple M_A at the free end (see figure). The lengths of the overhang and the main span are a and L, respectively.

Determine the angle of rotation θ_A and deflection δ_A at end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-10 Overhanging beam *ABC*



M_A = couple acting at the free end A (corresponding to the angle of rotation θ_A)

P = fictitious load corresponding to the deflection δ_A

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AB

$$M_{AB} = -M_A - Px \quad (0 \le x \le a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = -1 \qquad \frac{\partial M_{AB}}{\partial P} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT BC

Reaction at support C: $R_C = \frac{M_A}{L} + \frac{Pa}{L}$ (downward)

$$M_{BC} = -R_C x = -\frac{M_A x}{L} - \frac{Pax}{L} \quad (0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial M_A} = -\frac{x}{L} \qquad \frac{\partial M_{BC}}{\partial P} = -\frac{ax}{L}$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_A

$$\theta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_A}\right) dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-1)\,dx + \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{x}{L}\right) dx$$

Set $P = 0$:

$$\theta_A = \frac{1}{EI}\int_0^a M_A dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{x}{L}\right) dx$$

$$= \frac{M_A}{3EI}(L + 3a) \quad \text{(counterclockwise)} \quad \leftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_A

$$\delta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-x)dx + \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{ax}{L}\right) dx$$

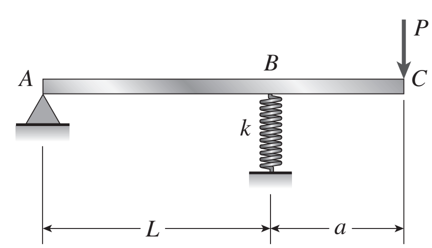
Set $P = 0$:

$$\delta_A = \frac{1}{EI}\int_0^a M_A x dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{ax}{L}\right) dx$$

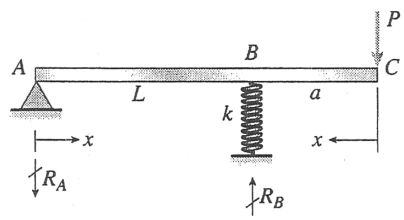
$$= \frac{M_A a}{6EI}(2L + 3a) \quad \text{(downward)} \quad \leftarrow$$

Problem 9.9-11 An overhanging beam ABC rests on a simple support at A and a spring support at B (see figure). A concentrated load P acts at the end of the overhang. Span AB has length L, the overhang has length a, and the spring has stiffness k.

Determine the downward displacement δ_C of the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-11 Beam with spring support



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$R_B = \frac{P}{L}(L + a) \quad \text{(upward)}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AB

$$M_{AB} = -R_A x = -\frac{Pax}{L} \qquad \frac{dM_{AB}}{dP} = -\frac{ax}{L}$$

$$(0 \le x \le L)$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT BC

$$M_{BC} = -Px \quad \frac{dM_{BC}}{dP} = -x \quad (0 \le x \le a)$$

STRAIN ENERGY OF THE SPRING (EQ. 2-38a)

$$U_S = \frac{R_B^2}{2k} = \frac{P^2(L + a)^2}{2kL^2}$$

STRAIN ENERGY OF THE BEAM (EQ. 9-80a)

$$U_B = \int \frac{M^2 dx}{2EI}$$

TOTAL STRAIN ENERGY U

$$U = U_B + U_S = \int \frac{M^2 dx}{2EI} + \frac{P^2(L + a)^2}{2kL^2}$$

APPLY CASTIGLIANO'S THEOREM (EQ. 9-87)

$$\delta_C = \frac{dU}{dP} = \frac{d}{dP}\int \frac{M^2 dx}{2EI} + \frac{d}{dP}\left[\frac{P^2(L + a)^2}{2kL^2}\right]$$

$$= \frac{d}{dP}\int \frac{M^2 dx}{2EI} + \frac{P(L + a)^2}{kL^2}$$

DIFFERENTIATE UNDER THE INTEGRAL SIGN (MODIFIED CASTIGLIANO'S THEOREM)

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{dM}{dP}\right)dx + \frac{P(L + a)^2}{kL^2}$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{Pax}{L}\right)\left(-\frac{ax}{L}\right)dx$$

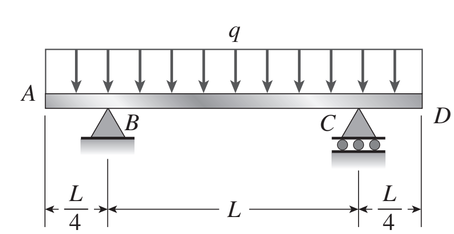
$$+ \frac{1}{EI}\int_0^a (-Px)(-x)\,dx + \frac{P(L + a)^2}{kL^2}$$

$$= \frac{Pa^2L}{3EI} + \frac{Pa^3}{3EI} + \frac{P(L + a)^2}{kL^2}$$

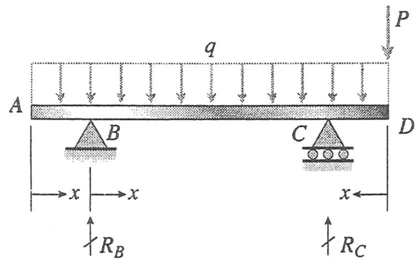
$$\delta_C = \frac{Pa^2(L + a)}{3EI} + \frac{P(L + a)^2}{kL^2} \qquad \leftarrow$$

Problem 9.9-12 A symmetric beam $ABCD$ with overhangs at both ends supports a uniform load of intensity q (see figure).

Determine the deflection δ_D at the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-12 Beam with overhangs



q = intensity of uniform load

P = fictitious load corresponding to the deflection δ_D

$\frac{L}{4}$ = length of segments AB and CD

L = length of span BC

$$R_B = \frac{3qL}{4} - \frac{P}{4} \qquad R_C = \frac{3qL}{4} + \frac{5P}{4}$$

Bending moments and partial derivatives

Segment AB

$$M_{AB} = -\frac{qx^2}{2} \qquad \frac{\partial M_{AB}}{\partial P} = 0 \qquad \left(0 \le x \le \frac{L}{4}\right)$$

Segment BC

$$M_{BC} = -\left[q\left(x + \frac{L}{4}\right)\right]\left[\frac{1}{2}\left(x + \frac{L}{4}\right)\right] + R_B x$$

$$= -\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3\,qL}{4} - \frac{P}{4}\right)x$$

$$(0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial P} = -\frac{x}{4}$$

Segment CD $\quad M_{CD} = -\frac{qx^2}{2} - Px \quad \left(0 \le x \le \frac{L}{4}\right)$

$$\frac{\partial M_{CD}}{\partial P} = -x$$

Modified Castigliano's theorem for deflection δ_D

$$\delta_D = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(0)dx$$

$$+ \frac{1}{EI}\int_0^{L}\left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3qL}{4} - \frac{P}{4}\right)x\right]$$

$$\times \left[-\frac{x}{4}\right]dx + \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2} - Px\right)(-x)dx$$

Set $P = 0$:

$$\delta_D = \frac{1}{EI}\int_0^{L}\left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \frac{3qL}{4}x\right]\left[-\frac{x}{4}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(-x)dx$$

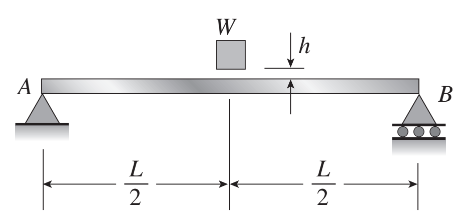
$$= -\frac{5qL^4}{768EI} + \frac{qL^4}{2048EI} = -\frac{37qL^4}{6144EI}$$

(Minus means the deflection is opposite in direction to the fictitious load P.)

$$\therefore \delta_D = \frac{37qL^4}{6144EI} \quad \text{(upward)} \qquad \leftarrow$$

Deflections Produced by Impact

The beams described in the problems for Section 9.10 have constant flexural rigidity EI. Disregard the weights of the beams themselves, and consider only the effects of the given loads.



Problem 9.10-1 A heavy object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure).

Obtain a formula for the maximum bending stress σ_{max} due to the falling weight in terms of h, σ_{st}, and δ_{st}, where σ_{st} is the maximum bending stressand δ_{st} is the deflection at the midpoint when the weight W acts on the beam as a statically applied load.

Plot a graph of the ratio σ_{max}/σ_{st} (that is, the ratio of the dynamic stress to the static stress) versus the ratio h/δ_{st}. (Let h/δ_{st} vary from 0 to 10.)

Solution 9.10-1 Weight *W* dropping onto a simple beam

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

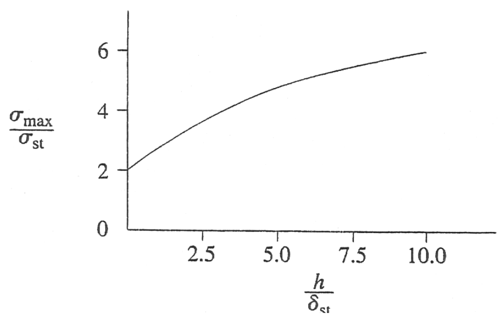
MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right] \quad \leftarrow$$

GRAPH OF RATIO σ_{max}/σ_{st}

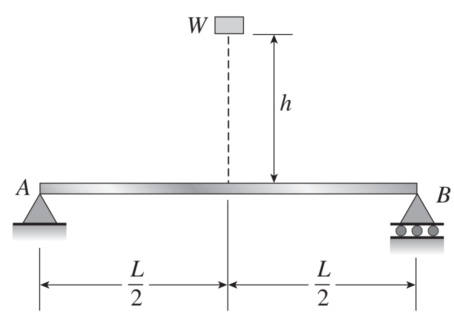


$\frac{h}{\delta_{st}}$	$\frac{\sigma_{max}}{\sigma_{st}}$
0	2.00
2.5	3.45
5.0	4.33
7.5	5.00
10.0	5.58

NOTE: $\delta_{st} = \dfrac{WL^3}{48EI}$ for a simple beam with a load at the midpoint.

Problem 9.10-2 An object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure). The beam has a rectangular cross section of area A.

Assuming that h is very large compared to the deflection of the beam when the weight W is applied statically, obtain a formula for the maximum bending stress σ_{max} in the beam due to the falling weight.



Solution 9.10-2 Weight W dropping onto a simple beam

Height h is very large.

MAXIMUM DEFLECTION (EQ. 9-95)

$$\delta_{max} = \sqrt{2h\delta_{st}}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = \sqrt{\frac{2h}{\delta_{st}}}$$

$$\sigma_{max} = \sqrt{\frac{2h\sigma_{st}^2}{\delta_{st}}} \quad (1)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \qquad \sigma_{st}^2 = \frac{W^2L^2}{16S^2}$$

$$\delta_{st} = \frac{WL^3}{48EI} \qquad \frac{\sigma_{st}^2}{\delta_{st}} = \frac{3WEI}{S^2L} \quad (2)$$

For a RECTANGULAR BEAM (with b, depth d):

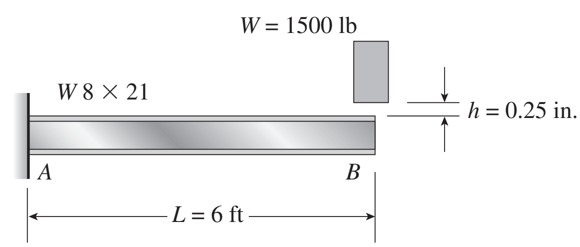
$$I = \frac{bd^3}{12} \qquad S = \frac{bd^2}{6} \qquad \frac{I}{S^2} = \frac{3}{bd} = \frac{3}{A} \quad (3)$$

Substitute (2) and (3) into (1):

$$\sigma_{max} = \sqrt{\frac{18WhE}{AL}} \quad \leftarrow$$

Problem 9.10-3 A cantilever beam AB of length $L = 6$ ft is constructed of a W 8×21 wide-flange section (see figure). A weight $W = 1500$ lb falls through a height $h = 0.25$ in. onto the end of the beam.

Calculate the maximum deflection δ_{max} of the end of the beam and the maximum bending stress σ_{max} due to the falling weight. (Assume $E = 30 \times 10^6$ psi.)



Solution 9.10-3 Cantilever beam

DATA: $L = 6$ ft $= 72$ in. $\quad W = 1500$ lb

$h = 0.25$ in. $\quad E = 30 \times 10^6$ psi

W 8×21 $\quad I = 75.3$ in.^{4} $\quad S = 18.2$ in.^{3}

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance: Simple beam with load at midpoint:

$$k = \frac{48EI}{L^3}$$

Cantilever beam with load at the free end: $k = \dfrac{3EI}{L^3}$

For the cantilever beam in this problem:

$$\delta_{st} = \frac{WL^3}{3EI} = \frac{(1500 \text{ lb})(72 \text{ in.})^3}{3(30 \times 10^6 \text{ psi})(75.3 \text{ in.}^4)}$$

$$= 0.08261 \text{ in.}$$

Equation (9-94):

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2} = 0.302 \text{ in.} \quad \leftarrow$$

MAXIMUM BENDING STRESS

Consider a cantilever beam with load P at the free end:

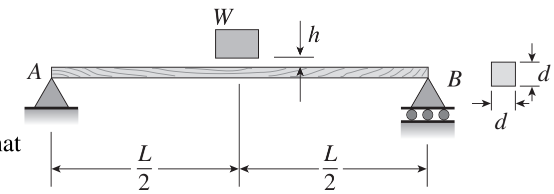
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{PL}{S} \qquad \delta_{max} = \frac{PL^3}{3EI}$$

Ratio: $\dfrac{\sigma_{max}}{\delta_{max}} = \dfrac{3\,EI}{SL^2}$

$$\therefore \sigma_{max} = \frac{3EI}{SL^2}\delta_{max} = 21{,}700 \text{ psi} \quad \leftarrow$$

Problem 9.10-4 A weight $W = 20$ kN falls through a height $h = 1.0$ mm onto the midpoint of a simple beam of length $L = 3$ m (see figure). The beam is made of wood with square cross section (dimension d on each side) and $E = 12$ GPa.

If the allowable bending stress in the wood is $\sigma_{allow} = 10$ MPa, what is the minimum required dimension d?



Solution 9.10-4 Simple beam with falling weight W

DATA: $W = 20$ kN $\quad h = 1.0$ mm $\quad L = 3.0$ m

$E = 12$ GPa $\quad \sigma_{allow} = 10$ MPa

CROSS SECTION OF BEAM (SQUARE)

d = dimension of each side

$$I = \frac{d^4}{12} \qquad S = \frac{d^3}{6}$$

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \left(\frac{WL}{4}\right)\left(\frac{6}{d^3}\right) = \frac{3WL}{2d^3} \qquad (2)$$

$$\delta_{st} = \frac{WL^3}{48EI} = \frac{WL^3}{48E}\left(\frac{12}{d^4}\right) = \frac{WL^3}{4Ed^4} \qquad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1)

$$\frac{2\sigma_{max}d^3}{3WL} = 1 + \left(1 + \frac{8hEd^4}{WL^3}\right)^{1/2}$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{2(10 \text{ MPa})d^3}{3(20 \text{ kN})(3.0 \text{ m})} = 1 + \left[1 + \frac{8(1.0 \text{ mm})(12 \text{ GPa})d^4}{(20 \text{ kN})(3.0 \text{ m})^3}\right]^{1/2}$$

$$\frac{1000}{9}d^3 - 1 = \left[1 + \frac{1600}{9}d^4\right]^{1/2} \quad (d = \text{meters})$$

SQUARE BOTH SIDES, REARRANGE, AND SIMPLIFY

$$\left(\frac{1000}{9}\right)^2 d^3 - \frac{1600}{9}d - \frac{2000}{9} = 0$$

$$2500d^3 - 36d - 45 = 0 \quad (d = \text{meters})$$

SOLVE NUMERICALLY

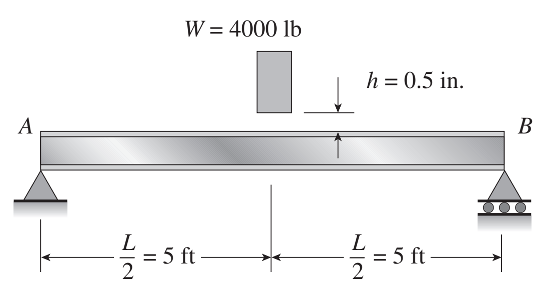
$d = 0.2804$ m $= 280.4$ mm

For minimum value, round upward.

$\therefore d = 281$ mm $\quad \leftarrow$

Problem 9.10-5 A weight $W = 4000$ lb falls through a height $h = 0.5$ in. onto the midpoint of a simple beam of length $L = 10$ ft (see figure).

Assuming that the allowable bending stress in the beam is $\sigma_{allow} = 18{,}000$ psi and $E = 30 \times 10^6$ psi, select the lightest wide-flange beam listed in Table E-1a in Appendix E that will be satisfactory.



Solution 9.10-5 Simple beam of wide-flange shape

DATA: $W = 4000$ lb $\quad h = 0.5$ in.

$L = 10$ ft $= 120$ in.

$\sigma_{allow} = 18{,}000$ psi $\quad E = 30 \times 10^6$ psi

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

or
$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \qquad \delta_{st} = \frac{WL^3}{48EI}$$

$$\frac{\sigma_{max}}{\sigma_{st}} = \sigma_{allow}\left(\frac{4S}{WL}\right) = \frac{4\sigma_{allow}S}{WL} \qquad (2)$$

$$\frac{2h}{\delta_{st}} = 2h\left(\frac{48EI}{WL^3}\right) = \frac{96hEI}{WL^3} \qquad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1):

$$\frac{4\sigma_{allow}S}{WL} = 1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}$$

REQUIRED SECTION MODULUS

$$S = \frac{WL}{4\sigma_{allow}}\left[1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}\right]$$

SUBSTITUTE NUMERICAL VALUES

$$S = \left(\frac{20}{3}\text{ in.}^3\right)\left[1 + \left(1 + \frac{5I}{24}\right)^{1/2}\right] \qquad (4)$$

(S = in.^{3}; I = in.^{4})

PROCEDURE

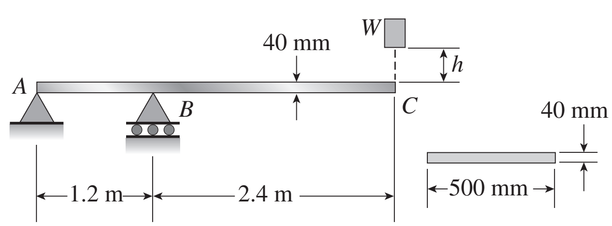
1. Select a trial beam from Table E-1a.
2. Substitute I into Eq. (4) and calculate required S.
3. Compare with actual S for the beam.
4. Continue until the lightest beam is found.

Trial beam	Actual I	Actual S	Required S
W 8 × 35	127	31.2	41.6 (NG)
W 10 × 45	248	49.1	55.0 (NG)
W 10 × 60	341	66.7	63.3 (OK)
W 12 × 50	394	64.7	67.4 (NG)
W 14 × 53	541	77.8	77.8 (OK)
W 16 × 31	375	47.2	66.0 (NG)

Lightest beam is W 14 × 53 ←

Problem 9.10-6 An overhanging beam ABC of rectangular cross section has the dimensions shown in the figure. A weight $W = 750$ N drops onto end C of the beam.

If the allowable normal stress in bending is 45 MPa, what is the maximum height h from which the weight may be dropped? (Assume $E = 12$ GPa.)



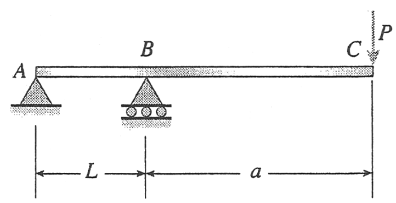
Solution 9.10-6 Overhanging beam

DATA: $W = 750\text{ N}$ $\quad L_{AB} = 1.2\text{ m.}$ $\quad L_{BC} = 2.4\text{ m}$

$E = 12\text{ GPa}$ $\quad \sigma_{\text{allow}} = 45\text{ MPa}$

$$I = \frac{bd^3}{12} = \frac{1}{12}(500\text{ mm})(40\text{ mm})^3$$
$$= 2.6667 \times 10^6\text{ mm}^4$$
$$= 2.6667 \times 10^{-6}\text{ m}^4$$
$$S = \frac{bd^2}{6} = \frac{1}{6}(500\text{ mm})(40\text{ mm})^2$$
$$= 133.33 \times 10^3\text{ mm}^3$$
$$= 133.33 \times 10^{-6}\text{ m}^3$$

DEFLECTION δ_C AT THE END OF THE OVERHANG



P = load at end C

L = length of span AB

a = length of overhang BC

From the answer to Prob. 9.8-5 or Prob. 9.9-3:

$$\delta_C = \frac{Pa^2(L + a)}{3EI}$$

Stiffness of the beam: $k = \dfrac{P}{\delta_C} = \dfrac{3EI}{a^2(L + a)}$ (1)

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

Simple beam with load at midpoint: $k = \dfrac{48EI}{L^3}$

Cantilever beam with load at free end: $k = \dfrac{3EI}{L^3}$ Etc.

For the overhanging beam in this problem (see Eq. 1):

$$\delta_{st} = \frac{W}{k} = \frac{Wa^2(L + a)}{3EI} \qquad (2)$$

in which $a = L_{BC}$ and $L = L_{AB}$:

$$\delta_{st} = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{3EI} \qquad (3)$$

EQUATION (9-94):

$$\delta_{\max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

or

$$\frac{\delta_{\max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (4)$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{\max}}{\sigma_{st}} = \frac{\delta_{\max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (5)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL_{BC}}{S} \qquad (6)$$

MAXIMUM HEIGHT h

Solve Eq. (5) for h:

$$\frac{\sigma_{\max}}{\sigma_{st}} - 1 = \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\left(\frac{\sigma_{\max}}{\sigma_{st}}\right)^2 - 2\left(\frac{\sigma_{\max}}{\sigma_{st}}\right) + 1 = 1 + \frac{2h}{\delta_{st}}$$

$$h = \frac{\delta_{st}}{2}\left(\frac{\sigma_{\max}}{\sigma_{st}}\right)\left(\frac{\sigma_{\max}}{\sigma_{st}} - 2\right) \qquad (7)$$

Substitute δ_{st} from Eq. (3), σ_{st} from Eq. (6), and σ_{allow} for $\sigma_{\max}$:

$$h = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI}\left(\frac{\sigma_{\text{allow}}S}{WL_{BC}}\right)\left(\frac{\sigma_{\text{allow}}S}{WL_{BC}} - 2\right) \qquad (8)$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (8):

$$\frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI} = 0.08100\text{ m}$$

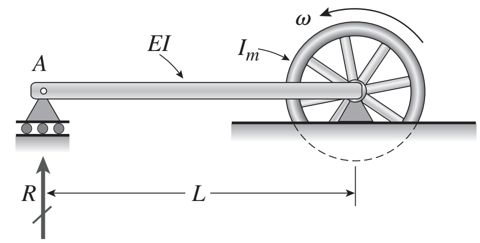
$$\frac{\sigma_{\text{allow}}S}{WL_{BC}} = \frac{10}{3} = 3.3333$$

$$h = (0.08100\text{ m})\left(\frac{10}{3}\right)\left(\frac{10}{3} - 2\right) = 0.36\text{ m}$$

or $h = 360\text{ mm}$ $\leftarrow$

Problem 9.10-7 A heavy flywheel rotates at an angular speed ω (radians per second) around an axle (see figure). The axle is rigidly attached to the end of a simply supported beam of flexural rigidity EI and length L (see figure). The flywheel has mass moment of inertia I_m about its axis of rotation.

If the flywheel suddenly freezes to the axle, what will be the reaction R at support A of the beam?



Solution 9.10-7 Rotating flywheel

NOTE: We will disregard the mass of the beam and all energy losses due to the sudden stopping of the rotating flywheel. Assume that *all* of the kinetic energy of the flywheel is transformed into strain energy of the beam.

KINETIC ENERGY OF ROTATING FLYWHEEL

$$\text{KE} = \frac{1}{2} I_m \omega^2$$

STRAIN ENERGY OF BEAM $\quad U = \int \frac{M^2 dx}{2EI}$

$M = Rx$, where x is measured from support A.

$$U = \frac{1}{2EI}\int_0^L (Rx)^2 dx = \frac{R^2L^3}{6EI}$$

CONSERVATION OF ENERGY

$$\text{KE} = U \qquad \frac{1}{2} I_m \mathsf{v}^2 = \frac{R^2L^3}{6EI}$$

$$R = \sqrt{\frac{3EI\, I_m \mathsf{v}^2}{L^3}} \qquad \leftarrow$$

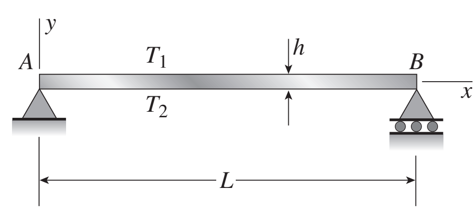
NOTE: The moment of inertia I_m has units of $\text{kg} \cdot \text{m}^2$ or $\text{N} \cdot \text{m} \cdot \text{s}^2$

Temperature Effects

The beams described in the problems for Section 9.11 have constant flexural rigidity EI. In every problem, the temperature varies linearly between the top and bottom of the beam.

Problem 9.11-1 A simple beam AB of length L and height h undergoes a temperature change such that the bottom of the beam is at temperature T_2 and the top of the beam is at temperature T_1 (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_A at the left-hand support, and the deflection $\delta_{\max}$ at the midpoint.



Solution 9.11-1 Simple beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)x}{h} + C_1$

B.C. 1 (Symmetry) $v'\left(\dfrac{L}{2}\right) = 0$

$\therefore\ C_1 = -\dfrac{\alpha L(T_2 - T_1)}{2h}$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h} - \dfrac{\alpha L(T_2 - T_1)x}{2h} + C_2$

B.C. 2 $v(0) = 0 \qquad \therefore\ C_2 = 0$

$v = -\dfrac{\alpha(T_2 - T_1)(x)(L - x)}{2h} \qquad \leftarrow$

(positive v is upward deflection)

$v' = -\dfrac{\alpha(T_2 - T_1)(L - 2x)}{2h}$

$\theta_A = -v'(0) = \dfrac{\alpha L(T_2 - T_1)}{2h} \qquad \leftarrow$

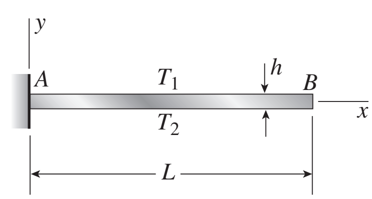
(positive θ_A is clockwise rotation)

$\delta_{max} = -v\left(\dfrac{L}{2}\right) = \dfrac{\alpha L^2(T_2 - T_1)}{8h} \qquad \leftarrow$

(positive δ_{max} is downward deflection)

Problem 9.11-2 A cantilever beam AB of length L and height h (see figure) is subjected to a temperature change such that the temperature at the top is T_1 and at the bottom is T_2.

Determine the equation of the deflection curve of the beam, the angle of rotation θ_B at end B, and the deflection δ_B at end B.



Solution 9.11-2 Cantilever beam with temperature differential

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)}{h}x + C_1$

B.C. 1 $v'(0) = 0 \qquad \therefore\ C_1 = 0$

$v' = \dfrac{\alpha(T_2 - T_1)}{h}x$

$v = \dfrac{\alpha(T_2 - T_1)}{h}\left(\dfrac{x^2}{2}\right) + C_2$

B.C. 2 $v(0) = 0 \qquad \therefore\ C_2 = 0$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h} \qquad \leftarrow$

(positive v is upward deflection)

$\theta_B = v'(L) = \dfrac{\alpha L(T_2 - T_1)}{h} \qquad \leftarrow$

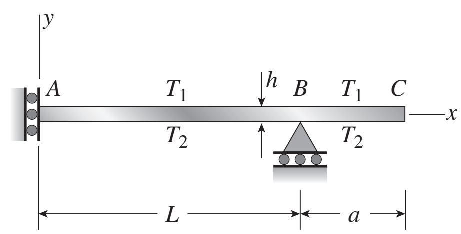
(positive θ_B is counterclockwise rotation)

$\delta_B = v(L) = \dfrac{\alpha L^2(T_2 - T_1)}{2h} \qquad \leftarrow$

(positive δ_B is upward deflection)

Problem 9.11-3 An overhanging beam ABC of height h has a guided support at A and a roller at B. The beam is heated to a temperature T_1 on the top and T_2 on the bottom (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_C at end C, and the deflection δ_C at end C.



Solution 9.11-3

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha(T_2 - T_1)}{h}$$

$$v' = \frac{\alpha(T_2 - T_1)x}{h} + C_1$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} + C_1x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(L) = 0 \qquad C_2 = -\dfrac{\alpha(T_2 - T_1)L^2}{2h}$

$$v(x) = \frac{\alpha(T_2 - T_1)(x^2 - L^2)}{2h}$$

$$\delta_C = v(L + a) = \frac{\alpha(T_2 - T_1)[(L + a)^2 - L^2]}{2h}$$

$$= \frac{\alpha(T_2 - T_1)(2La + a^2)}{2h} \quad \leftarrow$$

Upward

$$\theta_C = v'(L + a) = \frac{\alpha(T_2 - T_1)(L + a)}{h} \quad \leftarrow$$

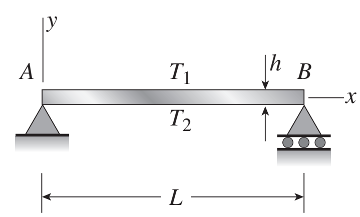
Counter Clockwise

Problem 9.11-4 A simple beam AB of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A; that is, assume the temperature difference varies *linearly* along the beam:

$$T_2 - T_1 = T_0 x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance.

(a) Determine the maximum deflection δ_{max} of the beam.
(b) Repeat for *quadratic* temperature variation along the beam, $T_2 - T_1 = T_0 x^2$.



Solution 9.11-4

(a) $(T_2 - T_1) = T_0 x$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x}{h}$$

$$v' = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1x + C_2$$

B.C. $v(0) = 0 \qquad C_2 = 0$

B.C. $v(L) = 0 \qquad C_1 = -\dfrac{\alpha T_0 L^2}{6h}$

$$v(x) = \frac{\alpha T_0(x^3 - L^2 x)}{6h}$$

$$v'(x) = \frac{\alpha T_0}{2h}\left(x^2 - \frac{L^2}{3}\right)$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0}{2h}\left(x^2 - \frac{L^2}{3}\right) \qquad x = \frac{L}{\sqrt{3}}$$

$$\delta_{max} = -v\left(\frac{L}{\sqrt{3}}\right)$$

$$= -\frac{\alpha T_0\left[\left(\frac{L}{\sqrt{3}}\right)^3 - L^2\frac{L}{\sqrt{3}}\right]}{6h}$$

$$\delta_{max} = \frac{\alpha T_0 L^3}{9\sqrt{3}h} \quad \text{Downward} \quad \leftarrow$$

(b) $(T_2 - T_1) = T_0\,x^2$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x^2}{h}$$

$$v' = \frac{\alpha T_0 x^3}{3h} + C_1$$

$$v = \frac{\alpha T_0 x^4}{12h} + C_1 x + C_2$$

B.C. $v(0) = 0 \qquad C_2 = 0$

B.C. $v(L) = 0 \qquad C_1 = -\frac{\alpha T_0 L^3}{12h}$

$$v(x) = \frac{\alpha T_0(x^4 - L^3 x)}{12h}$$

$$v'(x) = \frac{\alpha T_0}{3h}\left(x^3 - \frac{L^3}{4}\right)$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0}{3h}\left(x^3 - \frac{L^3}{4}\right) \qquad x = \frac{L}{\sqrt{2}}$$

$$\delta_{max} = -v\left(\frac{L}{\sqrt{2}}\right)$$

$$= -\frac{\alpha T_0\left[\left(\frac{L}{\sqrt{2}}\right)^4 - L^3\frac{L}{\sqrt{2}}\right]}{12h}$$

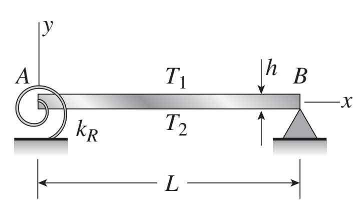
$$\delta_{max} = \frac{\alpha T_0 L^4}{48h}(2\sqrt{2} - 1) \quad \leftarrow$$

Downward

Problem 9.11-5 Beam AB, with elastic support k_R at A and pin support at B, of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A; that is, assume the temperature difference varies *linearly* along the beam:

$$T_2 - T_1 = T_0 x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance. Assume the spring at A is unaffected by the temperature change.



(a) Determine the maximum deflection δ_{max} of the beam.
(b) Repeat for *quadratic* temperature variation along the beam, $T_2 - T_1 = T_0\,x^2$.
(c) What is δ_{max} for (a) and (b) above if k_R goes to infinity?

Solution 9.11-5

(a) $(T_2 - T_1) = T_0 x$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x}{h}$$

$$v' = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(L) = 0 \qquad C_2 = -\frac{\alpha T_0 L^3}{6h}$

$$v(x) = \frac{\alpha T_0 (x^3 - L^3)}{6h}$$

$$v'(x) = \frac{\alpha T_0 x^2}{2h}$$

Maximum deflection

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0 x^2}{2h} \qquad x = 0$$

$$\delta_{max} = -v(0) = \frac{\alpha T_0 L^3}{6h} \quad \text{Downward} \quad \leftarrow$$

(b) $(T_2 - T_1) = T_0 x^2$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0 x^2}{h}$$

$$v' = \frac{\alpha T_0 x^3}{3h} + C_1$$

$$v = \frac{\alpha T_0 x^4}{12h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \qquad C_1 = 0$

B.C. $v(L) = 0 \qquad C_2 = -\frac{\alpha T_0 L^4}{12h}$

$$v(x) = \frac{\alpha T_0 (x^4 - L^4)}{12h}$$

$$v'(x) = \frac{\alpha T_0 x^3}{3h}$$

Maximum deflection

Set $v'(x) = 0$ and solve for x

$$0 = \frac{\alpha T_0 x^3}{3h} \qquad x = 0$$

$$\delta_{max} = -v(0) = \frac{\alpha T_0 L^4}{12h} \quad \text{Downward} \quad \leftarrow$$

(c) Changing k_R does not change δ_{max} in both cases.